KRULL DIMENSION FOR LIMIT GROUPS IV: ADJOINING ROOTS

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ABSTRACT. This is the fourth and last paper in a sequence on Krull dimension for limit groups, answering a question of Z. Sela. In it we finish the proof, analyzing limit groups obtained from other limit groups by adjoining roots. We generalize our work on Scott complexity and adjoining roots from the previous paper in the sequence to the category of limit groups.

1. INTRODUCTION, NOTATION, THEOREMS

It will take a moment to establish the notation and define the objects needed to state our main theorem. Roughly, we are interested in solutions, in the category of limit groups, to equations of the form "adjoin a root to g." We can give no specific characterizations of solutions, but under special circumstances arising in the second paper in this series, [Lou08b], we are able to show that most of the time solutions are unique.

The notation $Z_G(E)$ indicates the centralizer in G of a subgroup E. The set of images of edge groups incident to a vertex group V of a graph of groups decomposition is denoted by $\mathcal{E}(V)$. The phrase "'X' is controlled by 'Y'" should be read as "there is a function f, defined independently of 'X' and 'Y', such that $X \leq f(Y)$".

Let G be a group. A system of equations over G is a collection of words in the alphabet $\{x_i, g \mid g \in G\}$, where the x_i are variables distinct from the elements of G. The elements of G are the coefficients, and the coefficients occuring in Σ are the coefficients of Σ. If Σ is a system of equations over G there is a canonical group G_Σ associated to Σ with the presentation $\langle x_i, G \mid \Sigma\rangle$, where the x_i are the variables occuring in Σ. If the map $G \to G_\Sigma$ is injective then Σ has a solution. If $G < H$ and the inclusion map extends to G_Σ then Σ has a solution in H. In analogy with field extensions, suppose Σ is a system of equations over G. If $G < H$ and the inclusion map extends to a surjection $G_\Sigma \twoheadrightarrow H$ then H is a *splitting group* for Σ, and G is the *ground group*. Splitting groups are partially ordered by the relation "maps onto." Every pair $G < H$ is a ground-splitting pair for some (in general, many) system of equations $\Sigma(G, H)$. A tuple (G, H, G') is *flight* if H and G' are both splitting groups over G, and $H \twoheadrightarrow G'$.

2000 *Mathematics Subject Classification.* Primary: 20F65; Secondary: 20E05, 20E06.

Key words and phrases. limit group, krull dimension, JSJ, fully residually free.

Most of this research was done while at the University of Utah. The author also gratefully acknowledges support from the National Science Foundation, MSRI, and Rutgers University, Newark.

One may ask for splitting groups in a category $\mathcal{C}$ of groups. If $H \in \mathcal{C}$ is a splitting group, then H is a splitting group in $\mathcal{C}$. If $\mathcal{C}$ is the class of all groups, then there are maximal $\mathcal{C}$–splitting groups, but this is not the case for general classes.

A sequence of inclusions $\mathcal{G} = (\mathcal{G}(0) < \mathcal{G}(1) < \cdots)$ is a *tower*. A *staircase* is a pair of sequences $(\mathcal{G}, \mathcal{H})$ such that $\mathcal{G}$ is a tower and $\mathcal{G}(i)$ is a splitting group for (some system) $\Sigma(\mathcal{G}(i-1), \mathcal{H}(i))$, that is, $(\mathcal{G}(i-1), \mathcal{H}(i), \mathcal{G}(i))$ is a flight. All staircases considered in this paper have the property that all coefficients lie in $\mathcal{G}(0)$. The name staircase comes from the fact that a commutative diagram representing one looks like a staircase and walks up a tower.

Definition 1.1 (Adjoining roots)**.** Let G be a finitely generated group, $\mathcal{E}$ a collection of nontrivial abelian subgroups of G. For each $E \in \mathcal{E}$, let $\mathcal{F}(E)$ be a collection of finite index supergroups of E, with an inclusion map $i_{E,F} \colon E \hookrightarrow F$ for each $F \in \mathcal{F}(E)$, and let $\mathcal{F}(\mathcal{E})$ be the collection $\{\mathcal{F}(E)\}$. Let

$$G\left[\sqrt[\mathcal{F}(\mathcal{E})]{\mathcal{E}}\right] \coloneqq \langle G, F \mid E = i_{E,F}(E) \rangle_{F \in \mathcal{F}(E), E \in \mathcal{E}}$$

A finitely generated group H is obtained from G by *adjoining roots* $\mathcal{F}(\mathcal{E})$ *to* $\mathcal{E}$ if $G < H$ and the inclusion map extends to a surjection

$$G\left[\sqrt[\mathcal{F}(\mathcal{E})]{\mathcal{E}}\right] \twoheadrightarrow H$$

Let $\Sigma = \Sigma(\mathcal{E}, \mathcal{F}(\mathcal{E}))$ be a system of equations corresponding to the identification of E with $i_{E,F}(E)$ for all E and $F \in \mathcal{F}(E)$. Then H is a splitting group for Σ. We call H a cyclic extension of G because the relations are all of the form "adjoin a root to G."

Most of the time the specific nature of $\mathcal{F}$ is immaterial, and we usually eliminate it from the notation. To further compress the language used, sometimes we simply write that H is obtained from G by adjoining roots.

A group is *conjugately separated abelian*, or CSA, if maximal abelian subgroups are malnormal. Let $\sim_Z$ be the relation "is conjugate into the centralizer of". This is an equivalence relation as long as the group is CSA. Two important consequences of CSA are commutative transitivity and that every nontrivial abelian subgroup is contained in a unique maximal abelian subgroup.

Commutative transitivity can occasionally be used to simplify systems of equations. Suppose H is obtained from G by adjoining roots $\mathcal{F}(\mathcal{E})$ to $\mathcal{E}$. Let η be the inclusion map. We remove some redundancy by singling out a subcollection of each of $\mathcal{E}$ and $\mathcal{F}(\mathcal{E})$, and replacing each subcollection by a single element. Fix some $\sim_Z$ equivalence class $[E]$. By conjugating we may assume that each element of $[E]$ is a subgroup of $Z_G([E])$. Replace $[E]$ by $\{Z_G([E])\}$, and replace $\cup_{B \in [E]} \mathcal{F}(B)$ by

$$\langle Z_G([E]), F \mid B = i_{B,F}(B) \rangle^{\text{ab}}_{B \in [E], F \in \mathcal{F}(B)}$$

Then by commutative transitivity H is a quotient of

$$G\left[\sqrt[\mathcal{F}(\mathcal{E})]{\mathcal{E}}\right]$$

Since limit groups are CSA we make this reduction without comment. Since $\mathcal{F}(E)$ has a single element after this simplification, we will generally use the less ostentatious notation $F(E)$ or just $\sqrt{E}$. We will call a system of equations without any such redundancy *reduced*.

Definition 1.2 (Staircase)**.** A *cyclic staircase* is a staircase, with tower $\mathcal{G}$, equipped with a family families $\mathcal{E}$ of subgroups $\mathcal{E}_i$ of $\mathcal{G}(i)$, $(\mathcal{G}, \mathcal{H}, \mathcal{E})$, such that

- $(\mathcal{G}(i-1), \mathcal{H}(i), \mathcal{G}(i))$ is a flight; $\mathcal{H}(i)$ is obtained from $\mathcal{G}(i-1)$ by adjoining roots to $\mathcal{E}_{i-1}$
- Each $E' \in \mathcal{E}_i$ in $\mathcal{G}(i)$ centralizes, up to conjugacy, the image of an element E of $\mathcal{E}_{i-1}$. If $E \in \mathcal{E}_{i-1}$ is mapped to $E' \in \mathcal{E}_i$ then we require that the image of $Z_G(E)$ in $Z_{G'}(E')$ be finite index.

To fix notation, the maps $\mathcal{G}(i) \hookrightarrow \mathcal{G}(i+1)$, $\mathcal{G}(i) \hookrightarrow \mathcal{H}(i+1)$, and $\mathcal{H}(i+1) \twoheadrightarrow \mathcal{G}(i+1)$ are denoted by η_i, ν_i, and π_{i+1}, respectively. The length of $\mathcal{G}$ is denoted $\|\mathcal{G}\|$.

It will be handy to have a rough description of a staircase. A staircase of limit groups is

- *freely decomposable* if all $\mathcal{G}(i)$ are freely decomposable
- *freely indecomposable* if all $\mathcal{G}(i)$ are freely indecomposable
- QH–*free* if no $\mathcal{G}(i)$ has a QH subgroup
- *mixed* if it has both freely decomposable and freely indecomposable groups, or, if freely indecomposable, has both groups with and without QH subgroups. Otherwise it is *pure*.

Definition 1.3. Let (i_j) strictly increasing sequence of indices. A staircase $(\mathcal{V}, \mathcal{W})$, such that $\mathcal{V}(j) = \mathcal{G}(i_j)$ and $\mathcal{W}(j) = \mathcal{H}(i_j)$, with maps obtained by composing maps from $(\mathcal{G}, \mathcal{H})$, is a *contraction* of $(\mathcal{G}, \mathcal{H})$, and is *based on* (i_j).

To see that a contraction of a cyclic staircase is a staircase consider the following diagram:

$$\begin{tikzcd}
 & \mathcal{H}(i_j+1) \arrow[d, two heads] & \cdots & \mathcal{H}(i_{j+1}-1) \arrow[d, two heads] & \mathcal{H}(i_{j+1}) \arrow[d, two heads] \\
\mathcal{G}(i_j) \arrow[ur, hook] \arrow[r, hook] & \mathcal{G}(i_j+1) & \cdots & \mathcal{G}(i_{j+1}-1) \arrow[ur, hook] \arrow[r, hook] & \mathcal{G}(i_{j+1})
\end{tikzcd}$$

Each $E \in \mathcal{E}_i$ has finite index image in its counterpart in $\mathcal{E}_{i+1}$, the image of E in its counterpart in $\mathcal{E}_{i_{j+1}}$ is finite index. Extending an abelian group by a finite index super-group multiple times can be accomplished by extending once.

The need for contractions explains the restriction that each $E \in \mathcal{E}_i$ contain a conjugate of the image of some $E' \in \mathcal{E}_{i-1}$. If this is not the case, then there is no hope for the existence of contractions; we can't adjoin a root to an element that isn't there.

A *segment* of a staircase is a contraction whose indices are consecutive, that is $i_{j+1} - i_j = 1$ for all j.

Let $\mathcal{E}$ be a collection of elements of a CSA group G. We denote by $\|\mathcal{E}\|$ the number of $\sim_Z$ equivalence classes in $\mathcal{E}$. The *complexity* of $(\mathcal{G}, \mathcal{H}, \mathcal{E})$ is the triple $\mathrm{Comp}((\mathcal{G}, \mathcal{H}, \mathcal{E})) := (\mathrm{b}_1(\mathcal{G}), \mathrm{depth}_{\mathsf{pc}}(\mathcal{H}), \|\mathcal{E}\|)$. Complexities are not compared lexicographically: $(b', d', e') \leq (b, d, e)$ if $b' \leq b$, $d' \leq d$, and $e' \leq e + 2(d - d')b$. That this defines a partial order follows easily from the definition. The inequality is strict if one of the coordinate inequalities is strict. See Definition 2.4 and the material thereafter for a discussion of depth. Another immediate consequence of the definition of $\leq$ is that it is locally finite.[1]

Let $(\mathcal{G}, \mathcal{H}, \mathcal{E})$ be a staircase. The quantity $\mathrm{NInj}((\mathcal{G}, \mathcal{H}, \mathcal{E}))$ is the number of indices i such that $\mathcal{H}(i) \twoheadrightarrow \mathcal{G}(i)$ is *not* an isomorphism.

Theorem 1.4. *Let $(\mathcal{G}, \mathcal{H}, \mathcal{E})$ be a staircase. There is a function $\mathrm{NInj}(\mathrm{Comp}((\mathcal{G}, \mathcal{H}, \mathcal{E})))$ such that*

$$\mathrm{NInj}((\mathcal{G}, \mathcal{H}, \mathcal{E})) \leq \mathrm{NInj}(\mathrm{Comp}((\mathcal{G}, \mathcal{H}, \mathcal{E})))$$

Remark 1.5. Although it would be nice to assign a complexity $c()$ to a limit group such that if, in a flight (G, H, G'), $c(G) = c(G')$, then $H \twoheadrightarrow G'$ is an isomorphism, this doesn't seem possible, and the approach taken in this paper requires that complexities be computed and compared in context.

Acknowledgments. The author thanks Mladen Bestvina, Mark Feighn, and Zlil Sela for many discussions related to this paper.

2. Complexities of sequences

The main object which enables this analysis of adjoining roots is the JSJ decomposition, a device for encoding families of splittings of groups. This exposition borrows from [BF03, RS97]. A GAD, or *generalized abelian decomposition* of a group G is a finite graph of groups decomposition over abelian edge groups such that every vertex group is marked as one of *rigid*, *abelian*, or QH, where by QH we mean is the fundamental group of a compact surface with boundary possessing two intersecting essential simple closed curves. Moreover, edge groups adjacent to a QH vertex group must be conjugate to boundary components of the surface. If A is an abelian vertex group, the *peripheral* subgroup of A is the subgroup of A which dies under every map $A \to \mathbb{Z}$ killing all incident edge groups.

We say that two GAD's of a limit group are *equivalent* if they have the same elliptic subgroups. A splitting is *visible* in a GAD Δ if it corresponds to cutting a QH vertex group along a simple closed curve, a one-edged splitting of an abelian vertex group in which the peripheral subgroup is elliptic, or is a one edged splitting corresponding to an edge from an equivalent decomposition. If Δ is a GAD, then $g \in G$ is Δ–elliptic if elliptic in every one-edged splitting of G visible in Δ. Let $\mathrm{Ell}(\Delta)$ be the set of Δ–elliptic elements.

Let G be a freely indecomposable finitely generated group, and let $\mathcal{C}$ be a family of one-edged splittings of G such that

- edge groups are abelian,

[1] Fix a and b. Then $\{x \mid a \leq x \leq b\}$ is finite.

- noncyclic abelian subgroups are elliptic.

The main construction of JSJ theory is that given a family of splittings $\mathcal{C}$ satisfying these conditions, there is a GAD Δ such that $\mathrm{Ell}(\Delta) = \cap_{C \in \mathcal{C}} \mathrm{Ell}(C)$.

An abelian JSJ decomposition of G is a GAD $\mathrm{AJSJ}(G)$ such that the set of AJSJ–elliptic elements corresponds to the collection of all one-edged splittings satisfying the bullets above. The existence of a JSJ decomposition is somewhat subtle, as one needs to bound the size of a GAD arising in this way [Sel01, Theorem 3.9]. If G is a nonelementary freely indecomposable limit group then G has a nontrivial JSJ decomposition. If G is elementary, the JSJ is a point.

In this paper we are interested in the principle cyclic JSJ decomposition, which is the JSJ associated to the family of principle cyclic splittings.

Definition 2.1 ([Sel01])**.** A one-edged splitting over a cyclic subgroup is *inessential* if at least one vertex group is cyclic, and is *essential* otherwise. A *principle cyclic* splitting of a limit group is an essential one-edged splitting $G \cong A *_C B$ or $G \cong A*_C$, over a cyclic subgroup C, such that either $Z_G(C)$ is cyclic or A is abelian.

The *principle cyclic* JSJ of a freely indecomposable limit group is the JSJ decomposition corresponding to the family of principle cyclic splittings. We denote the principle cyclic JSJ by $\mathrm{JSJ}(G)$.

Let $\mathcal{E} \subset G$. The *principle cyclic* JSJ *of* G, *relative to* $\mathcal{E}$ is a JSJ decomposition corresponding to the family of all principle cyclic splittings of G such that each member of $\mathcal{E}$ is elliptic. We denote the relative JSJ by $\mathrm{JSJ}(G; \mathcal{E})$. A *principle cyclic decomposition* is simply a relative principle cyclic JSJ for some collection $\mathcal{E}$.

The *restricted principle cyclic* JSJ, or *restricted* JSJ for short, of a freely indecomposable limit group G *with* QH subgroups is the relative JSJ decomposition associated to the set of principle cyclic splittings whose edge groups are hyperbolic in some other principle cyclic splitting. It is obtained from the JSJ by collapsing all edges not adjacent to some QH vertex group. If G doesn't have QH vertex groups, then the restricted JSJ is just the principle cyclic JSJ. The restricted principle cyclic JSJ is denoted by $\mathrm{RJSJ}(G)$

That limit groups have principle cyclic splittings is [Sel01, Theorem 3.2]. It need not be the case that every splitting visible in the principle cyclic JSJ is principle; for instance, a boundary component of a QH vertex group may be the only edge attached to a cyclic vertex group. The splitting corresponding to the boundary component is not essential, but is certainly visible in the principle cyclic JSJ.

In this paper we work primarily with the principle cyclic JSJ of G, indicated by $\mathrm{JSJ}(G)$, and the RJSJ. If Δ is a graph of groups decomposition then T_Δ is the Bass-Serre tree corresponding to Δ.

We can give a more explicit description of the principle cyclic JSJ. Consider the abelian JSJ of a limit group G. Clearly all QH vertex groups of $\mathrm{AJSJ}(G)$ appear as vertex groups of $\mathrm{JSJ}(G)$. If A is an abelian vertex group of $\mathrm{AJSJ}(G)$ with noncyclic peripheral subgroup, since there is no principle cyclic splitting of G over a subgroup of A, the subgroup of G generated by A and conjugates of rigid

vertex groups having nontrivial intersection with A must be elliptic in $\mathrm{JSJ}(G)$. If R is a rigid vertex group of G and an edge group E incident to R has noncyclic centralizer in R, then the subgroup of G generated by R and any conjugate of a rigid vertex group R' intersecting R in a nontrivial subgroup of E is also elliptic. From this we see that the principle cyclic JSJ of G must have the following form:

- Every abelian vertex group has cyclic peripheral subgroup. If R is adjacent to an abelian vertex group A, E the edge group, then R does not have an essential one-edged splitting over E in which each element of $\mathcal{E}(R)$ is elliptic.
- If an edge e incident to a rigid vertex group R has noncyclic centralizer in R, then the edge is attached to a boundary component of a QH vertex group.
- If two edges incident to a rigid vertex group R have the same centralizer in R, then they are both incident to QH vertex groups, and their centralizer in R is noncyclic.

The JSJ decomposition of a limit group, be it abelian or principle cyclic, is only unique up to morphisms of graphs of groups preserving elliptic subgroups. Some principle cyclic JSJ's are more convenient to work with than others, and we assume throughout that

- Edge groups not adjacent to QH vertex groups are closed under taking roots, and edge maps of edge groups into QH vertex groups are isomorphisms with the corresponding boundary components.
- There are no inessential splittings visible in the JSJ, other than from valence one cyclic vertex groups attached to boundary components of QH vertex groups.

Let R be a rigid vertex group of the full abelian JSJ of a limit group G, and let $\bar{R}$ be the subgroup of G generated by R and all elements with powers in R. A decomposition with the properties above can be thought of as the JSJ decomposition associated to the family of principle cyclic splittings in which all $\bar{R}$, R a vertex group of the abelian JSJ, are elliptic.

In general, there are infinitely many principle cyclic decompositions of a limit group, all obtained from the principle cyclic JSJ by folding, cutting QH vertex groups along simple closed curves, and collapsing subgraphs.

Lemma 2.2. *Let G be a limit group and $\mathcal{E} \subset G$ a collection of elements of G. Then there are at most $2^{\|\mathcal{E}\|}$ equivalence classes of principle cyclic decompositions in which some elements of $\mathcal{E}$ are elliptic.*

Proof. If $E \in \mathcal{E}$ is elliptic, then so is any $E' \in \mathcal{E}$ such that $E \sim_Z E'$. $\square$

We need to adapt the definition of the analysis lattice of a limit group given in [Sel01, §4] to the inductive proof given in section 4.4. A limit group is *elementary* if it is abelian, free, or the fundamental group of a closed surface.

Definition 2.3 (Principle cyclic analysis lattice)**.** The *principal cyclic analysis lattice* of a limit group G is the rooted tree of groups whose levels are defined as follows:

0: G
$\frac{1}{2}$: The free factors of a Grushko decomposition of G.
1: The vertex groups at level 1 are the vertex groups of the RJSJ.
$n\left(\frac{1}{2}\right)$: Rinse and repeat, incrementing the index by one each time.

If an elementary limit group is encountered, it is a terminal leaf of the tree.

Definition 2.4. The *depth* of a limit group H is the number of levels in its principle cyclic analysis lattice, and is denoted $\mathrm{depth}_{\mathrm{pc}}(H)$.

The *depth* of a staircase $(\mathcal{G}, \mathcal{H}, \mathcal{E})$ is $\max\left\{\mathrm{depth}_{\mathrm{pc}}(\mathcal{H}(i))\right\}$, and is is denoted $\mathrm{depth}_{\mathrm{pc}}((\mathcal{G}, \mathcal{H}, \mathcal{E}))$. The *first betti number* of $(\mathcal{G}, \mathcal{H}, \mathcal{E})$ is the first betti number of $\mathcal{G}(1)$.

It is not always necessary to refer to the family $\mathcal{E}$, so we suppress it from the notation when its size is irrelevant. That the depth is well defined is a consequence of Theorem 2.5.

Theorem 2.5. *The depth of the principle cyclic analysis lattice of a limit group L is controlled by its rank.*

Proof. We only need to worry about the possibility that the principle cyclic analysis lattice contains a long branch of the form $G_0 > G_1 > \cdots$, where each G_i is freely indecomposable, has no QH vertex groups, no noncyclic abelian vertex groups, and $\mathrm{JSJ}(G_i)$ has only one nonabelian vertex group G_{i+1}. After observing [Lou08a, Hou08] that L has a strict resolution of length at most $6\,\mathrm{rk}(L)$, the proof is identical to [Lou08b, Theorem 2.11]. □

We motivate our proof of Theorem 1.4 and the previous definition with an example.

Example 2.6. *Suppose that $\mathcal{G}(i)$ has a one-edged* JSJ *decomposition with two nonabelian vertices for all i. Since a limit group has a principle cyclic splitting, the one-edged splitting of $\mathcal{G}(i)$ must be of the form $\mathcal{G}_1(i) *_{\langle e_i\rangle} \mathcal{G}_2(i)$. By Lemma 3.1, if H is obtained from G by adjoining roots, then G acts hyperbolically in all splittings of H. In particular, every vertex group of H contains a vertex group of G. If the* JSJ *of H was a loop, then the map to the underlying graph kills G, but since the map $G \to H$ is almost onto on homology, this cannot happen. Thus each $\mathcal{H}(i)$ has a one-edged* JSJ *decomposition $\mathcal{H}_1(i) *_{\langle f_i\rangle} \mathcal{H}_2(i)$.*

The triple $\mathcal{G}(i-1) \hookrightarrow \mathcal{H}(i) \twoheadrightarrow \mathcal{G}(i)$ has the following form: The pairs $(\mathcal{G}_j(i-1), \langle e_{i-1}\rangle)$ map to the pairs $(\mathcal{G}_j(i), \langle e_i\rangle)$ and $(\mathcal{H}_j(i), \langle f_i\rangle)$, and the maps η_i, ν_i, and π_i respect these one-edged splittings. We'll show later that in fact $\mathcal{H}_j(i)$ is obtained from $\mathcal{G}_j(i-1)$ by adjoining roots. By work from [Lou08b] *$\mathcal{G}_j(i)$ is obtained from the image of $\mathcal{H}_j(i)$ by iteratively adjoining roots to the incident edge group (See Appendix A). Let $\mathcal{G}'_j(i)$ be the image of $\mathcal{H}_j(i)$ in $\mathcal{G}_j(i)$. Now consider the staircase $(\mathcal{G}'_j(i), \mathcal{H}_j(i))$. The sequence $\mathcal{H}_j$ has strictly lower depth than $\mathcal{H}$.*

By induction on Comp, *there is an upper bound on the number of indices such that $\mathcal{H}_j(i) \to \mathcal{G}(i)$ is not injective, and for at most twice that bound, both maps $\mathcal{H}_j(i) \to \mathcal{G}_j(i)$ are injective. Then $\mathcal{H}(i) \twoheadrightarrow \mathcal{G}(i)$ is strict for such indices. Since*

every Dehn twist in $\langle f_i \rangle$ *pushes forward to a Dehn twist in* $\langle e_{i+1} \rangle$, π_i *is an isomorphism.*

3. Aligning JSJ decompositions

Let G be a finitely generated group acting on a simplicial tree T minimally and without inversions. It is a standard fact that the quotient T/G is the underlying graph of a graph of groups decomposition of G. If $H < G$ is a finitely generated subgroup, there is a minimal subtree $S \subset T$ fixed (setwise) by H, and the action of H on S endows H with a graph of groups decomposition. Additionally, there is an induced map of quotient graphs $S/H \to T/G$.

We are interested in the following problem: Suppose G, H, T, and S are as above, G and H freely indecomposable limit groups, T the Bass-Serre tree corresponding to the principle cyclic JSJ of H. We say that G and H are aligned if $S/H \to T/G$ is an isomorphism of graphs and S/H is the underlying graph of the principle cyclic JSJ of H. Give a simple computable criterion which guarantees that G and H are aligned.

As long as $G^{ab} \to H^{ab}$ is virtually onto, we are able to answer this question in a reasonable way, constructing a (monotonically decreasing) complexity, equality of which will guarantee alignment of JSJs. The properties of the alignment are then used to construct graphs of spaces and maps between them which resemble Stallings' immersions. The main idea of this section is that an inclusion as above must either "tighten up" the Grushko/JSJ becoming simpler in a quantifiable way, or can be written as a map of graphs of groups respecting the JSJ decompositions.

Let T be the Bass-Serre tree corresponding to the principle cyclic JSJ of H, let S be the minimal subtree for G. The quotient S/G is finite and it follows from the definitions that the induced graphs of groups decomposition of G is principle. For convenience, we usually conflate underlying graphs and graphs of groups decompositions. Let $\Delta_H = T/H$ and $\Delta_G = S/G$ be the underlying graphs, and let $\eta_\#$ be the induced map. We label a vertex v of Δ_G by the corresponding label on $\eta_\#(v)$, unless G_v is abelian, in which case we label it abelian anyway. The map $\eta_\#$ is well behaved:

- If v is rigid then the edge groups adjacent to v have nonconjugate centralizers in G_v unless they are all attached to boundary components of QH vertex groups.
- Let B be a maximal connected subgraph of Δ_G such that every vertex is abelian. Commutative transitivity implies that G_B is abelian, and the fact that all noncyclic abelian subgroups of H are elliptic in T_H implies that B is a tree.
- If v is abelian and $\eta_\#(v)$ is nonabelian, then $\eta_\#(v)$ is rigid.
- A valence one cyclic v is adjacent to a QH w. This follows from the assumption that the only edge groups of H allowed to be not closed under taking roots are adjacent to QH vertices.

Lemma 3.1. *Let $\eta\colon G \to H$ be a homomorphism of freely indecomposable limit groups such that $H^1(H,\mathbb{Z}) \to H^1(G,\mathbb{Z})$ is injective. Then G is hyperbolic in every essential one-edged abelian splitting of H.*

If R is a nonabelian vertex group of a GAD Δ_H *of H, then G intersects a conjugate of R in a nonabelian subgroup. If Δ_G is the induced decomposition of G, and there is only one nonabelian vertex group R' of Δ_G mapping to R, then the map on underlying graphs is a submersion at R'.*

Proof. Claim: If G acts elliptically in some essential one-edged splitting then there is a map $H \twoheadrightarrow \mathbb{Z}$ which kills G. If the one-edged splitting is an HNN extension the claim is clear. If not, then both vertex groups of the amalgam have a map onto $\mathbb{Z}$ which kills the incident edge group.

To see the second half, suppose not, and let Δ'_H be the decomposition of H obtained by conjugating edge maps to R so that all incident edges either have the same or nonconjugate centralizers, and folding together edges of the conjugated decomposition which have the centralizers. Then pull all the centralizers of incident edge groups across the edge they centralize. If T is the tree for Δ'_H, and S is the minimal G–invariant subtree, then the map $S/G \to T/H$ clearly misses the vertex corresponding to R. Let Δ'' be the decomposition of H obtained by collapsing all edges not adjacent to R. Then G is elliptic in Δ''. The first part provides a contradiction.

If the map is not a submersion on the level of Δ_H, then the of graphs of groups $\Delta'_G \to \Delta'_H$ is not a submersion onto R either, and there is an edge incident to R missed by Δ'_G. This edge represents an essential splitting of H, and so we again have a contradiction. $\square$

Definition 3.2 (Complexity of JSJs)**.** Let G be a finitely generated freely indecomposable limit group with principle cyclic decomposition $G = \Delta(\mathcal{R}, \mathcal{Q}, \mathcal{A}, \mathcal{E})$, where each $R \in \mathcal{R}$ is rigid, each $Q \in \mathcal{Q}$ is QH, each $A \in \mathcal{A}$ is finitely generated abelian, and each $E \in \mathcal{E}$ is an infinite cyclic edge group. Let

- $c_q(G) := |\sum_{Q\in\mathcal{Q}} \chi(Q)|$ is the total Euler characteristic of QH subgroups.
- $c_{bq}(G) := \sum_{Q\in\mathcal{Q}} \#\partial Q$ is the total number of boundary components of QH vertex groups
- $\mathcal{Z}(G)$ is the collection of conjugacy classes of centralizers of edge groups of G. Warning: *not* the center of G.
- For a given rigid vertex R of the principle cyclic JSJ decomposition, let $v(R)$ be the valence of R. This is the same as the number of conjugacy classes of centralizers of incident edge groups *in* R.
- $c_a(G) := \sum_{A\in\mathcal{A}} (\mathrm{rk}(A) - 1)$
- $c_b(G) = c_a(G) + \mathrm{b}_1(\Delta)$

The *complexity* of G with respect to Δ is the ordered tuple

$$\mathrm{JComp}(G,\Delta) = (c_q(G), -c_{bq}(G), |\mathcal{Z}|, c_b(G), \mathrm{b}_1(\Delta), |\mathcal{R}|, \sum_{R\in\mathcal{R}} v(R))$$

The “,Δ” is suppressed from the notation if Δ is the principle cyclic JSJ of G.

Complexities are compared lexicographically. The complexity JComp_i is the restriction of JComp to the first i coordinates.

Throughout this section G and H are freely indecomposable limit groups, $\eta\colon G \hookrightarrow H$, and $\eta^{\#}\colon H^1(H,\mathbb{Z}) \to H^1(G,\mathbb{Z})$ is injective.

We need to be able to compare the complexity of a principle decomposition to the complexity of the JSJ.

Lemma 3.3. *Let G be a freely indecomposable limit group with principle cyclic* JSJ Δ_G, *let $\mathcal{E}$ be a fixed family of subgroups of G, and let Δ be the principle cyclic decomposition of G associated to the family of principle cyclic splittings in which each $E \in \mathcal{E}$ is elliptic. Then $\mathrm{JComp}(G,\Delta) \leq \mathrm{JComp}(G)$, with equality if and only if Δ is the* JSJ.

Proof. We can construct Δ by cutting QH vertex groups of Δ_G along simple closed curves, folding, and collapsing subgraphs. To handle c_b, observe that any collection of disjoint simple closed curves on QH vertex groups of Δ can be completed to a collection which achives at most $c_b(G)$.

The inequalities on c_q and c_{bq} are obvious, and if they are equal, then the identity map simply identifies QH vertex groups. The remaining inequalities are obvious. □

We spread the proof of Theorem 3.6 across the next two lemmas.

Lemma 3.4. *$c_q(G) \geq c_q(H)$. If equality holds then $c_{bq}(G) \leq c_{bq}(H)$.*

Proof. Let T be the Bass-Serre tree for the restricted JSJ of H. Since η is injective, G inherits a graph of groups decomposition Δ from its action on T. Let Q be a vertex group of Δ conjugate into some element Q' of $\mathcal{Q}(H)$. There are two possibilities: Q either has finite or infinite index in Q'. If Q has infinite index and is nontrivial then G must be freely decomposable, contrary to hypothesis. Thus Q is either trivial or finite index.

Let c be a simple closed curve on some element Q' of $\mathcal{Q}(H)$ giving a essential one-edged splitting Δ_c of H. By Lemma 3.1 G acts hyperbolically in Δ_c, hence there is some Q which maps to a finite index subgroup of a conjugate of Q'. The graph of groups decomposition Δ of G is obtained by slicing QH vertex groups of G along simple closed curves, folding, and collapsing subgraphs of the resulting decomposition. This immediately gives $c_q(G) \geq c_q(H)$.

Suppose equality holds. Let c_k be the simple closed curves cutting the QH vertex groups of $\mathrm{JSJ}(G)$, and let $Q'_1, \dots, Q'_m$ be the complementary components which don't map to QH vertex groups of H. Since $c_q(G) = c_q(H)$, each component Q'_j has Euler characteristic 0. Any such complementary component cannot be boundary parallel, thus if there are any then $c_{bq}(H) > c_{bq}(G)$. If equality holds then the QH subgroups of G and those of H are in one to one correspondence and the respective maps are isomorphisms. □

An inclusion $G \hookrightarrow H$ as above is QH–*preserving* if it is a one-to-one correspondence on QH vertex groups and the maps are isomorphisms. If H has an

inessential one-edged splitting Δ, then Δ corresponds to an edge connecting a valence one cyclic vertex group of JSJ(H) to a QH vertex group. If $G \hookrightarrow H$ is QH–preserving then it is necessarily bijective on such valence one vertex groups.

It follows immediately from Lemma 3.1 that if $G \hookrightarrow H$ is QH–preserving then $|\mathcal{Z}(G)| \geq |\mathcal{Z}(H)|$.

Lemma 3.5. $\mathrm{JComp}_5(G) \geq \mathrm{JComp}_5(H)$. *If equality holds then there is an induced bijection* $\mathcal{A}(G) \to \mathcal{A}(H)$, *and for each* A, $A/P(A) \to \eta_{\#}(A)/P(\eta_{\#}(A))$ *is virtually onto.*

Proof. We first handle c_b.

Let Δ_H be the principle cyclic JSJ of H, and let Δ_G be the decomposition G inherits from its action on T_{Δ_H}. We may assume that $G \hookrightarrow H$ is QH–preserving, is bijective on conjugacy classes of centralizers of edge groups. Let

$$H_1(H; \Delta_H^{(0)} \setminus \mathcal{A}(H)) = H_1(\Delta_H) \oplus \bigoplus_{A \in \mathcal{A}(H)} A/P(A)$$

Similarly, define $H_1(G; \Delta_G^{(0)} \setminus \mathcal{A}(G))$. The composition $G \to H \to H_1(H; \Delta_H^{(0)} \setminus \mathcal{A}(H))$ factors through $H_1(G; \Delta_G^{(0)} \setminus \mathcal{A}(G))$. Since $H^1(H, \mathbb{Z}) \hookrightarrow H^1(G, \mathbb{Z})$ the map $H_1(G; \Delta_G^{(0)} \setminus \mathcal{A}(G)) \to H_1(H; \Delta_H^{(0)} \setminus \mathcal{A}(H))$ must be virtually onto. But $\mathrm{rk}(H_1(H; \Delta_H^{(0)} \setminus \mathcal{A}(H))) = c_b(H)$ and $\mathrm{rk}(H_1(G; \Delta_G^{(0)} \setminus \mathcal{A}(G))) \leq c_b(G)$.

Let Δ be an essential one-edged splitting of H in which all QH subgroups are elliptic. Let T be the corresponding Bass-Serre tree. By Lemma 3.1 G doesn't fix a point in T and it inherits an essential splitting Δ' from this action. Since η is bijective of the sets of QH subgroups, and restricts to isomorphisms between them, every QH vertex group of G acts elliptically in T. Thus there is an edge group E' of JSJ(G) which maps to a conjugate of the edge group of Δ. Furthermore, E' is an essential splitting, otherwise G acts elliptically in Δ.

Let $A \in \mathcal{A}(G)$ be a noncyclic abelian vertex group. If no element of $\mathcal{A}(H)$ contains the image of A, then $c_b(G) > c_b(H)$. If equality holds there is a well defined map $\mathcal{A}(G) \to \mathcal{A}(H)$.

Let A be an abelian vertex group of G, and $\eta_{\#}(A)$ the associated vertex group of H. Since $H^1(H) \to H^1(G)$ is injective, the map $A/P(A) \oplus H_1(\Gamma_G) \to \eta_{\#}(A)/P(\eta_{\#}(A)) \oplus H_1(\Gamma_H)$ must be virtually onto. This map sends $A/P(A)$ to $\eta_{\#}(A)/P(\eta_{\#}(A))$ hence $\mathrm{b}_1(\Delta_G) \geq \mathrm{b}_1(\Delta_H)$, and if $\mathrm{b}_1(\Delta_G) = \mathrm{b}_1(\Delta_H)$ then $A/P(A) \to \eta_{\#}(A)/P(\eta_{\#}(A))$ must be virtually onto. $\square$

Theorem 3.6. $\mathrm{JComp}_6(G) \geq \mathrm{JComp}_6(H)$. *If* $\mathrm{JComp}_6(G) = \mathrm{JComp}_6(H)$ *then*

$$\sum_{R \in \mathcal{R}(G)} v(R) \geq \sum_{R \in \mathcal{R}(H)} v(R),$$

i.e., $\mathrm{JComp}(G) \geq \mathrm{JComp}(H)$. *If* $\eta\colon G \hookrightarrow H$, $\mathrm{JComp}(G) = \mathrm{JComp}(H)$, *then* η *is bijective on vertex and edge groups, maps abelian vertex, edge, and peripheral subgroups to finite index subgroups of their respective images. The map from the underlying graph of the* JSJ *of* G *to the underlying graph of the* JSJ *of* H *is an isomorphism.*

The number of values the complexity can take is controlled by b_1.

Proof of Theorem 3.6. Assume $\mathrm{JComp}_5(G) = \mathrm{JComp}_5(H)$. By Lemma 3.5, the inclusion is a one-to-one correspondence on noncyclic abelian vertex groups.

Let Δ_H be the principle cyclic JSJ of H, let $\pi\colon \Delta_G \to \Delta_H$ be the induced map of underlying graphs , and let R be nonabelian non-QH vertex group of Δ_H. By Lemma 3.1 there is a nonabelian vertex group of Δ_G which maps to R. Since η is bijective on QH subgroups, there is a rigid vertex group R' of G which maps to R. If $\mathrm{JComp}_5(G) = \mathrm{JComp}_5(H)$ then R' is the unique such vertex group.

Again, by Lemma 3.1, since there is only one vertex group R' mapping to R, the map $\mathcal{E}(R') \to \mathcal{E}(R)$ is onto and $v(R') \geq v(R)$.

Let Z be an essential cyclic abelian vertex group of Δ_H, and let $Z_1, \ldots, Z_k$ be the vertex groups of Δ_G mapping to Z. Since η is bijective on nonabelian vertex groups, and since all vertex groups adjacent to Z are nonabelian, the induced map $\eta_\#\colon \sqcup \mathcal{E}(Z_i) \to \mathcal{E}(Z)$ is bijective. Arguing as in Lemma 3.1, $k = 1$ and the map $\mathcal{E}(Z_1) \to \mathcal{E}(Z)$ is bijective. The same observation shows that if A is noncyclic abelian, then there is a unique A' mapping to A and that the map on the link is onto. The map is also injective, again because η is a bijective on nonabelian vertex groups.

Thus, if the complexities are equal, then the inclusion must induce a homeomorphism of underlying graphs. By construction, the map is label preserving, and it automatically respect all incidence and conjugacy data from the respective JSJ decompositions.

This shows that $\mathrm{JComp}(G, \Delta_G) \geq \mathrm{JComp}(H)$, and if equality holds, then the morphism $\Delta_G \to \Delta_H$ is of the correct form. By Lemma 3.3 $\mathrm{JComp}(G) \geq \mathrm{JComp}(G, \Delta_G)$, and if $\mathrm{JComp}(G) = \mathrm{JComp}(H)$, then Δ_G is just the principle cyclic JSJ of G. This gives the first half of the theorem.

The bound on the number of values the complexity can take follows from either acylindrical accessibility [Sel97] plus the bound on the rank of a limit group with complexity b_0, or [Lou08b, Lemma 2.7], which gives a bound on the complexity of the principle cyclic JSJ in terms of the first betti number. Those arguments bound the number of essential vertex groups. Adjoining roots doesn't increase the first betti number, so if b_1 and b_2 are boundary components of a QH vertex group adjacent to inessential vertex groups, then a simple closed curve cutting off a pair of pants with b_1, b_2 as the two other boundary components makes a contribution of one to $\mathrm{b}_1(G)$; n nonintersecting simple closed curves as above make a contribution of n to $\mathrm{b}_1(G)$, thus each QH vertex group is attached to at most $2\,\mathrm{b}_1(G)$ inessential vertex groups. Since $\mathrm{b}_1(G)$ controls the number of QH vertex groups, there are boundedly many inessential abelian vertex groups. $\square$

In light of Theorem 3.6, if $\mathrm{JComp}(G) = \mathrm{JComp}(H)$, then we say that G and H are *aligned*. Before representing injections of limit groups topologically, we devote a section to proving Theorem 1.4, assuming the material from section 5.

4. Proof of Theorem 1.4

The bound implicitly computed in the proof of Theorem 1.4 can be made slightly better if we show that nonabelian limit groups with first betti number 2 are free. The next lemma is not necessary, but we record it here for lack of a better place to put it. In [FGM+98], Fine, et al., classify limit groups with rank at most three. The next lemma shows that in rank two the rank can be relaxed to first betti number.

Lemma 4.1. *Let G be a limit group with first betti number 2. Then $G \cong \mathbb{F}_2$ or $\mathbb{Z}^2$.*

Proof. We may assume G is nonabelian and freely indecomposable. If G is abelian it satisfies the theorem trivially, and if freely decomposable, the free factors are limit groups with first betti number one, and must be infinite cyclic.

The proof is by induction on the depth of the cyclic analysis lattice. All essential cyclic splittings of G are HNN extensions, otherwise there is a one-edged cyclic splitting such that each vertex group has betti number at least two, and G therefore has first betti number at least three. By a simple variation of the proof of Theorem 2.5 the depth of the cyclic analysis lattice of G is finite. Suppose that G has a QH vertex group Q. Then any essential simple closed curve on Q must correspond to an HNN extension of G: $G = G'*_E$. Since the splitting comes from a QH vertex group, G' must be freely decomposable, hence is $\mathbb{F}_2$. If G has no QH vertex groups it's principle cyclic JSJ decomposition must be a bouquet of circles. Let $G = G_0 > G_1 > G_2 > \cdots > G_n$ be a sequence of vertex groups of cyclic JSJ decompositions such that G_i, $i < n-1$, is freely indecomposable and has a bouquet of circles as its principle cyclic JSJ, terminating at the first index n such that such that G_n is freely decomposable, hence free. This chain must have finite length since the cyclic analysis lattice is finite. We argue that G_n free implies that G_{n-1} is free.

Let $f\colon G_{n-1} \to \mathbb{F}$ be a homomorphism such that $f(G_n)$ has nonabelian image. Since G_{n-1} is an HNN extension of G_n, by Corollary 1.6 of [Lou08c], the images of the incident edge groups in G_n can be conjugated to a basis for G_n and G_{n-1} is freely decomposable, contrary to hypothesis. □

Definition 4.2 (Extension)**.** An *extension* of a pure staircase $(\mathcal{G}, \mathcal{H})$ is a staircase $(\mathcal{G}, \mathcal{H}')$ such that the diagrams in Figure 1 commute. An extension is *admissible* if one of the following mutually exclusive conditions holds.

- $\mathcal{G}$ is freely decomposable, and the freely indecomposable free factors of $\mathcal{H}'(i)$ embed in $\mathcal{H}(i)$ under σ_i.
- $\mathcal{G}$ is freely indecomposable, has QH subgroups, and the vertex groups of the decomposition of $\mathcal{H}'(i)$ obtained by collapsing all edges not adjacent to QH vertex groups embed in $\mathcal{H}(i)$ for all i. (This is just the restricted principle cyclic JSJ.)
- $\mathcal{G}$ is freely indecomposable, QH–free, and for all i, vertex groups of the (restricted) principle cyclic JSJ of $\mathcal{H}'(i)$ embed in vertex groups of $\mathcal{H}(i)$ under σ_i.

An admissible extension has the property that each σ_i is strict, surjective, and maps elliptic subgroups of a decomposition of $\mathcal{H}'(i+1)$ to elliptic subgroups of

$$\begin{tikzcd}
 & \mathcal{H}'(i+1) \arrow[dr, two heads, "\sigma_i"] \arrow[ddr, two heads, "\pi'_i"'] & \\
\mathcal{G}(i)\left[\sqrt{\mathcal{E}_i}\right] \arrow[ur, two heads] & & \mathcal{H}(i+1) \arrow[d, two heads, "\pi_i"] \\
\mathcal{G}(i) \arrow[u, hook] \arrow[uur, hook, "\nu'_i"] \arrow[urr, hook, "\nu_i"] \arrow[rr, "\eta_i"] & & \mathcal{G}(i+1)
\end{tikzcd}$$

FIGURE 1. Extensions of sequences

$\mathcal{H}(i+1)$. The relation "mapps onto" partially orders the collection of extensions, and if $\mathcal{H}''$ is an extension of $\mathcal{H}'$ then $\mathcal{H}'' \geq_{\sqrt{}} \mathcal{H}'$. For some i, if σ_i is not one-to-one on the sets of vertex groups or edge groups then the inequality is strict. The envelope of a rigid vertex group of the principle cyclic JSJ is just the vertex group, hence if σ_i is one-to-one on the sets of vertex groups and edge groups then it is an isomorphism. If $(\mathcal{G}, \mathcal{H}'') \twoheadrightarrow (\mathcal{G}, \mathcal{H}') \twoheadrightarrow (\mathcal{G}, \mathcal{H})$ is a pair of admissible extensions then $(\mathcal{G}, \mathcal{H}'') \twoheadrightarrow (\mathcal{G}, \mathcal{H})$ is an admissible extension.

We work with staircases which are maximal with respect to $\geq_{\sqrt{}}$, rather than arbitrary staircases. To do this we have to pay a penalty, but not too large of one.

Lemma 4.3. *For all K there exists $C = C(K, \mathrm{Comp}(\,))$ such that if $(\mathcal{G}, \mathcal{H}, \mathcal{E})$ is a staircase and $\mathrm{NInj}(\mathrm{Comp}((\mathcal{G}, \mathcal{H}, \mathcal{E}))) = C(K, \mathrm{Comp}((\mathcal{G}, \mathcal{H}, \mathcal{E})))$, then there is a $\geq_{\sqrt{}}$–maximal extension of a contraction $(\mathcal{G}', \mathcal{H}', \mathcal{E}')$ of $(\mathcal{G}, \mathcal{H}, \mathcal{E})$ with $\mathrm{NInj}((\mathcal{G}', \mathcal{H}', \mathcal{E}')) \geq K$ and $\mathrm{Comp}((\mathcal{G}', \mathcal{H}', \mathcal{E}')) \leq \mathrm{Comp}((\mathcal{G}, \mathcal{H}, \mathcal{E}))$.*

The constants in this lemma do not depend on $\|\mathcal{E}\|$, and its proof is formally identical to the proof of [Lou08b, Theorem 4.2]. To adapt the proof, we need to show that the strict resolutions arising in an extension have bounded length. This follows from [Lou08b, Lemma 2.7], bounding the rank of $\mathcal{H}(i)$ from above by a function of $\mathrm{Comp}((\mathcal{G}, \mathcal{H}))$, but a proof more in the spirit of this paper goes as follows: If $\mathcal{H}^{(n)}(i) \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{H}(i)$ is a strict resolution appearing in a sequence of extensions, then $\mathrm{JComp}(G) \geq \mathrm{JComp}(\mathcal{H}^{(m)}(i))$ (See Lemma 3.6 and Definition 3.2.), moreover, if $\mathcal{H}^{(m+1)}(i) \twoheadrightarrow \mathcal{H}^{(m)}(i)$ is not injective on sets of vertex or edge spaces, or collapses subsurface groups of QH vertex groups, the complexity must decrease. By Theorem 3.6 the number of values the complexity takes is controlled by b_1, and the resolutions have length controlled by $\mathrm{Comp}((\mathcal{G}, \mathcal{H}, \mathcal{E}))$.

Each pure kind of staircase is handled in turn over the next three subsections. In all cases the strategy is the same: either there is compatibility between (collapses of) RJSJ decompositions/Grushko factorizations, the complexity decreases, or proper extensions exist.

4.1. **Freely decomposable.** This is the most singular case in that the arguments work for nearly all finitely generated groups, not just limit groups.

The complexity for freely indecomposable groups is used to show that base sequences of freely indecomposable staircase can be divided into segments such that

the base groups of a segment have the same JSJ decompositions, in the sense of Theorem 3.6. There is a similar complexity for freely decomposable groups which accomplishes the same thing but with regard to Grushko decompositions. The following theorem from [Lou08c] shows how the complexity for freely decomposable groups is useful.

Definition 4.4 (Scott complexity)**.** Let G be a finitely generated group with Grushko decomposition $G = G_1 * \cdots * G_p * \mathbb{F}_q$. The *Scott complexity* of G is the lexicographically ordered pair $\mathrm{sc}(G) := (q-1, p)$.

The number of Scott complexities of limit groups with $\mathrm{b}_1 = b$ is bounded by b^3.

Theorem 4.5 (Scott complexity and adjoining roots to groups)**.** *Suppose that* $\phi \colon G \hookrightarrow H$ *and* H *is a quotient of* $G' = G\left[\sqrt[k_i]{\gamma_i}\right]$, $\boldsymbol{\gamma}_i$ *a collection of distinct conjugacy classes of indivisible elements of* G *such that* $\boldsymbol{\gamma}_i \neq \boldsymbol{\gamma}_j^{-1}$ *for all* i, j *and* $\gamma_i \in \boldsymbol{\gamma}_i$. *Then* $\mathrm{sc}(G) \geq \mathrm{sc}(H)$. *If equality holds and* H *has no* $\mathbb{Z}_2$ *free factors, then there are presentations of* G *and* H *as*

$$G \cong G_1 * \cdots * G_p * \mathbb{F}_q^G, \quad H \cong H_1 * \cdots * H_p * \mathbb{F}_q^H$$

a partition of $\{\boldsymbol{\gamma}_i\}$ *into subsets* $\boldsymbol{\gamma}_{j,i}$, $j = 0, \ldots, p$, $i = 1, \ldots, i_p$, *representatives* $\gamma_{j,i} \in G_j \cap \boldsymbol{\gamma}_{j,i}$, $i \geq 1$, $\gamma_{0,i} \in \mathbb{F}_q^G \cap \boldsymbol{\gamma}_{0,i}$, *such that with respect to the presentations of* G *and* H*:*

- $\phi(G_i) < H_i$
- $G_j\left[\sqrt[k_{j,i}]{\gamma_{j,i}}\right] \twoheadrightarrow H_j$
- $\phi(\mathbb{F}_q^G) < \mathbb{F}_q^H$
- $\mathbb{F}_q^G = \langle \gamma_{0,1} \rangle * \cdots * \langle \gamma_{0,i_0} \rangle * F$
- $\mathbb{F}_q^H = \langle \sqrt{\gamma_{0,1}} \rangle * \cdots * \langle \sqrt{\gamma_{0,i_0}} \rangle * F$
- $G' \cong G_1\left[\sqrt{\gamma_{1,i}}\right] * \cdots * G_p\left[\sqrt{\gamma_{p,i}}\right] * \langle \sqrt{\gamma_{0,1}} \rangle * \cdots * \langle \sqrt{\gamma_{0,i_0}} \rangle * F$

All homomorphisms are those suggested by the presentations, and the maps on F *are the identity.*

This is [Lou08c, Theorem 1.2].

Remark 4.6. Theorem 4.5 is stated in terms of adjoining roots to cyclic subgroups of a group, whereas Definition 1.1 refers to collections of abelian subgroups. This difference is immaterial to the discussion here since adjoining roots to a noncyclic abelian group can be accomplished by adjoining roots to a suitable collection of cyclic subgroups. By passing from a noncyclic abelian subgroup to cyclic subgroups, the measure $\| \ \|$ is unchanged.

Definition 4.7 (Free products)**.** Let $(\mathcal{G}_i, \mathcal{H}_i)$ be a collection of staircases on the same index set I. Then the graded free product $((*_i \mathcal{G}_i), (*_i \mathcal{H}_i))$, with the obvious maps, is also a sequence of adjunctions of roots.

Lemma 4.8. *Suppose Theorem 1.4 holds for all staircases with complexity less than* (b_0, d_0, e_0). *Then Theorem 1.4 holds for pure freely decomposable staircases of complexity* (b_0, d_0, e_0).

Proof. Let $(\mathcal{G},\mathcal{H},\mathcal{E})$ be a staircase with complexity (b_0,d_0,e_0). Since limit groups are torsion free, no $\mathcal{G}(i)$ has a $\mathbb{Z}_2$ free factor, and by Theorem 4.5 for all but $\mathrm{b}_1(\mathcal{G}(1))^3$ indices i_j, the subsequences $\mathcal{G}(i_j)\hookrightarrow\mathcal{G}(i_j+1)\hookrightarrow\cdots\hookrightarrow\mathcal{G}(i_{j+1}-1)$ can be decomposed into free products of freely indecomposable groups staircases. Moreover, elements of $\mathcal{E}_i$ are either part of a basis of a free free factor of $\mathcal{G}(i)$ or are conjugate into a freely indecomposable free factor of $\mathcal{G}(i)$. Write $\mathcal{G}(i)$ as the free product

$$\mathcal{G}(i)_1*\cdots*\mathcal{G}(i)_p*F_i$$

given by the lemma, where F_i is a free group of rank q and $\mathrm{sc}(\mathcal{G}(i))=(q-1,p)$ for all i. Let $\mathcal{E}_i^j$ be the subset of $\mathcal{E}_i$ consisting of elements conjugate into $\mathcal{G}(i)_j$, and rearrange indices so that $\mathcal{G}(i)_j$ maps to $\mathcal{G}(i+1)_j$ for all j. Let $\mathcal{E}_i^0$ be the elements of $\mathcal{E}$ which are conjugate into F_i. By Theorem 4.5 there are decompositions

$$\mathcal{G}(i)\left[\sqrt{\mathcal{E}_i}\right]\cong\left(\mathcal{G}(i)_1\left[\sqrt{\mathcal{E}_i^1}\right]*\cdots*\mathcal{G}(i)_p\left[\sqrt{\mathcal{E}_i^p}\right]\right)*F_i\left[\sqrt{\mathcal{E}_i^0}\right]$$

where the last factor is free. Let $\mathcal{H}(i+1)_j := \mathrm{Im}_{\mathcal{H}(i+1)}(\mathcal{G}(i)_j\left[\sqrt{\mathcal{E}_i^j}\right])$ The sequence $\mathcal{H}'$ defined by

$$\mathcal{H}'(i+1):=(*_j\mathcal{H}(i+1)_j)*F_i\left[\sqrt{\mathcal{E}_i^0}\right]$$

is an extension of $\mathcal{H}$. Then $(\mathcal{G},\mathcal{H}',\mathcal{E})$ splits as a free product, the freely indecomposable free factors of which are $(\mathcal{G}_j,\mathcal{H}'_j,\mathcal{E}^j)$. These free factors have strictly lower b_1 than $\mathcal{G}$, depth at most $d_0=\mathrm{depth}_{\mathrm{pc}}(\mathcal{H})$, hence have $\mathrm{NInj}((\mathcal{G}_j,\mathcal{H}'_j,\mathcal{E}^j))\leq\mathrm{NInj}(\mathrm{Comp}(b_0-1,d_0,e_0))=:B$. If $\|\mathcal{G}\|>B\cdot\mathrm{b}_1(\mathcal{G})\geq B\cdot p$, then, for some index l, the map $\mathcal{H}'(l)\twoheadrightarrow\mathcal{G}(l)$ is visibly an isomorphism. Since this map factors through $\mathcal{H}(l)$, $\mathcal{H}(l)\twoheadrightarrow\mathcal{G}(l)$ is an isomorphism as well. □

We finish this subsection by proving the base case of the induction. Let $(\mathcal{G},\mathcal{H},\mathcal{E})$ be a maximal staircase of complexity $(b,2,e)$. By the proof of Lemma 4.5, the staircase splits as a free product of freely indecomposable staircases $(\mathcal{G}_i,\mathcal{H}_i,\mathcal{E}^i)$, and such that each $\mathcal{H}_i(j)$ is elementary. If $\mathcal{G}_i$ is abelian, then clearly $\mathcal{H}_i(j)\twoheadrightarrow\mathcal{G}_i(j)$ is an isomorphism, and if nonabelian, $\mathcal{H}_i(j)$ is the fundamental group of a closed surface. Since $\mathcal{G}_i$ is freely indecomposable, it is also the fundamental group of a closed surface. Divide $\mathcal{G}_i$ into segments such that the Euler characteristic is constant on each segment. Then $\mathcal{G}_i(j)\to\mathcal{G}_i(j+1)$ is an isomorphism on each segment and $\mathcal{H}_i(j)$ is a trivial extension of $\mathcal{G}_i(j-1)$ for all j on each segment, thus $\mathcal{H}_i(j)\twoheadrightarrow\mathcal{G}_i(j)$ is an isomorphism.

4.2. **Freely indecomposable, QH.** Lemma 5.3 allows us to handle injections $G\hookrightarrow H$, $\mathrm{JComp}(G)=\mathrm{JComp}(H)$, and such that G has a QH subgroup.

Lemma 4.9. *Suppose Theorem 1.4 holds for all staircases with strictly lower complexity than* (b_0,d_0,e_0). *Then Theorem 1.4 holds for staircases with* QH *subgroups and complexity* (b_0,d_0,e_0).

The strategy is to find an extension $(\mathcal{G}, \mathcal{H}', \mathcal{E})$ of $(\mathcal{G}, \mathcal{H}, \mathcal{E})$ such that the QH subgroups of $\mathcal{H}'$ are the "same" as those from $\mathcal{G}$. See Figure 2. The group $\mathcal{H}(i)$ may be a total mess, but luckily it is a homomorphic image of a limit group which shares its restricted JSJ with $\mathcal{G}(i)$ and $\mathcal{G}(i+1)$.

To do this an auxiliary lemma which follows immediately from Lemma 5.3 is needed.

Lemma 4.10. *Let G' be obtained from G by adjoining roots to a collection of abelian subgroups $\mathcal{E}$. If $\mathrm{JComp}(G) = \mathrm{JComp}(G')$ then every element $E \in \mathcal{E}$ such that $[E : F(E)] > 1$ is conjugate into a non-*QH *vertex group of* $\mathrm{RJSJ}(G)$.

We use the immersion representing $G \hookrightarrow G'$ constructed in subsection 5.1.

Proof. Fix E as in the statement of the lemma. We are done if we show that E is elliptic in every one edged splitting of G obtained by cutting a QH subgroup along an essential simple closed curve which doesn't cut off a Möbius band.[2] Start with an immersion representing the RJSJ decompositions of G and G', and let Σ_Q be the surface which contains c. There is a unique element $\eta_{\#}(Q)$ containing the image of Q, and the map $Q \to \eta_{\#}(Q)$ is surjective. Since $Q \to \eta_{\#}(Q)$ is represented by a homeomorphism $\Sigma_Q \to \Sigma_{\eta_{\#}(Q)}$ there is a simple closed curve $\eta_{\#}(c)$ contained in $\Sigma_{\eta_{\#}(Q)}$ and a closed annular closed neighborhood A of c mapping homeomorphically to a neighborhood of $\eta_{\#}(c)$. Use these neighborhoods to construct new graphs of spaces Y_G and $Y_{G'}$ representing G and G' by regarding the annulus as a new edge space and collapsing all but the newly introduced edges. By construction, the map $Y_G \hookrightarrow Y_{G'}$ is an immersion. By Lemma 5.3, if some element of $\mathcal{E}$ crosses c, then c maps to a power of $\eta_{\#}(c)$. $\square$

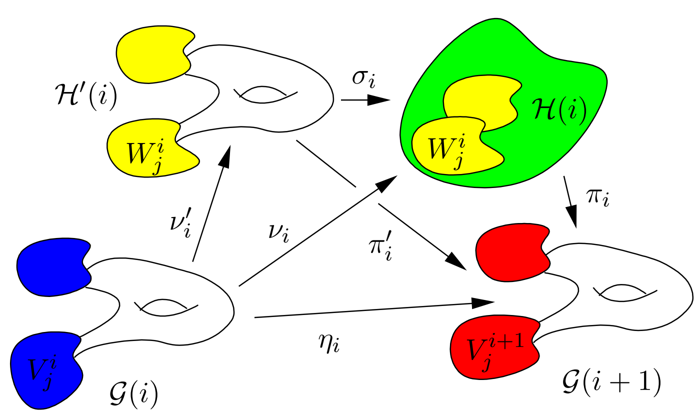

FIGURE 2. Illustration of Lemma 4.9

[2]We could have instead redefined an essential curve as one which gives a principle cyclic splitting and isn't boundary parallel.

Proof of Lemma 4.9. Suppose $(\mathcal{G}, \mathcal{H})$ is a staircase such that $\mathrm{sc}(\mathcal{G}(i))$ is the constant sequence and $c_q(\mathcal{G}(1)) \neq 0$. Let Δ_i be the RJSJ of $\mathcal{G}(i)$. Every edge of Δ_i is infinite cyclic and connects a vertex group to a boundary component of a QH vertex group. Since the inclusions $\mathcal{G}(i) \hookrightarrow \mathcal{G}(i+1)$ respected graphs of spaces, by the first part of Lemma 4.10, every element of $\mathcal{E}_i$ is conjugate into some non-QH vertex group of Δ_i. Let $V_1^i, \ldots, V_n^i$ be the non-QH vertex groups of Δ_i. We regard $\mathcal{G}(i)$ as a graph of groups $\Gamma(V_j^i, Q_k, E_l)$, where $\mathcal{G}(i) \hookrightarrow \mathcal{G}(i+1)$ is compatible with the decomposition Γ in the sense that V_j^i maps to a conjugate of V_j^{i+1}, the map respects edge group incidences, and the inclusion is the identity on the QH vertex groups Q_k.

Let $\mathcal{E}_i^j$ be the elements of $\mathcal{E}_i$ conjugate into V_j^i, and arrange that each $E \in \mathcal{E}_i^j$ is contained in V_j^i by conjugating if necessary. Let W_j^{i+1} be the image of $V_j^i\left[\sqrt{\mathcal{E}_i^j}\right]$ in $\mathcal{H}(i+1)$ and let $\mathcal{H}'(i+1) = \Gamma(W_j^{i+1}, Q_k, E_l)$. Then $(\mathcal{G}, \mathcal{H}', \mathcal{E})$ is an extension of $(\mathcal{G}, \mathcal{H}, \mathcal{E})$: The map implicit map $\sigma_i \colon \mathcal{H}'(i) \to \mathcal{H}(i)$ is clearly strict, therefore the sequence $\mathcal{H}'$ consists of limit groups. By definition, V_j^{i+1} is obtained from V_j^i by adjoining roots. Let $\mathcal{V}_j$ be the sequence $\mathcal{V}_j(i) = V_j^i$ and let $\mathcal{W}_j(i) = W_j^i$.

The staircases $(\mathcal{V}_j, \mathcal{W}_j, \mathcal{E}^j)$ all have lower first betti number than $(\mathcal{G}, \mathcal{H}, \mathcal{E})$. Let $B(b_0)$ be the maximal number of vertex groups of a limit group with first betti number b_0 [Lou08b, Lemma 2.7]. If $\|\mathcal{G}\| > \mathrm{NInj}((b_0 - 1, d_0, e_0)) \cdot B(b_0)$ then for at least one index l all $\mathcal{W}_j(l) \twoheadrightarrow \mathcal{V}_j(l+1)$ are injective. Thus π_l' is $\mathrm{Mod}(\mathcal{H}'(l), \text{RJSJ})$ strict. Since all modular automorphisms of $\mathcal{H}'(l)$ are either inner, Dehn twists in boundary components of QH vertex groups, or induced by boundary respecting homeomorphisms of surfaces representing QH vertex groups, by construction, every element of $\mathrm{Mod}(\mathcal{H}'(l), \text{RJSJ})$ pushes forward to a modular automorphism of $\mathcal{G}(l)$. An easy exercise shows that $\mathcal{H}'(l) \twoheadrightarrow \mathcal{G}(l)$ is an isomorphism. Since $\pi_l' = \pi_l \circ \sigma_l$, π_l is an isomorphism. $\square$

4.3. **Freely indecomposable, no QH.** The neighborhood of a vertex group V of a graph of groups decomposition is the subgroup generated by V and conjugates of adjacent vertex groups which intersect V nontrivially, and is denoted $\mathrm{Nbhd}(V)$. Let (G, H, G') be a flight and suppose G is freely indecomposable, has no QH vertex groups, and that $\mathrm{JComp}(G) = \mathrm{JComp}(G')$. Let $\eta \colon G \to G'$ be the inclusion map. An abelian vertex group A of G is *H–elliptic* if H doesn't have a principle cyclic splitting over a subgroup of $Z_H(\nu(A))$.

Let $\mathcal{A}_H$ be the collection of abelian vertex groups of G which are H–elliptic. Suppose that H is obtained from G by adjoining roots to the collection $\mathcal{E}$. Let $\mathcal{E}_H^{ell}$ be the sub-collection of $\mathcal{E}$ consisting of elements of E which are hyperbolic in the principle cyclic JSJ of G but which have elliptic image in the principle cyclic JSJ of H. Let $\mathrm{JSJ}_H(G)$ be the JSJ decomposition of G with respect to the collection of principle cyclic splittings in which all $\mathrm{Nbhd}(A)$, $A \in \mathcal{A}_H$, and $E \in \mathcal{E}_H^{ell}$ are elliptic:

$$\mathrm{JSJ}_H(G) := \mathrm{JSJ}(G; \left\{\mathrm{Nbhd}(A), E \mid A \in \mathcal{A}_H, E \in \mathcal{E}_H^{ell}\right\})$$

Let $\mathrm{JSJ}_H^*(G')$ be the JSJ decomposition of G' associated to the collection of all principle cyclic splittings of G' in which all $\eta_\#(A_H)$, $A \in \mathcal{A}_H$, and $\eta_\#(E)$, $E \in \mathcal{E}_H^{ell}$, are elliptic. That is

$$\mathrm{JSJ}_H^*(G') := \mathrm{JSJ}(G'; \left\{ \mathrm{Nbhd}(\eta_\#(A)), \eta_\#(E) \mid A \in \mathcal{A}_H, E \in \mathcal{E}_H^{ell} \right\})$$

The main lemma is that the decompositions of G and G' induced by H are intimately related to the principle cyclic JSJ of H as long as the flight admits no proper extensions. Let V be a vertex group of $\mathrm{JSJ}_H(G)$. There is a vertex group $\eta_\#(V)$ of $\mathrm{JSJ}_H^*(G')$ which contains the image of V. Let $\mathcal{E}_V$ be the collection of elements of $\mathcal{E}$ which are conjugate into V, along with the collection of incident edge groups. Likewise for $\eta_\#(V)$, let $\mathcal{E}(\eta_\#(V))$ be the set of centralizers of images of elements of $\mathcal{E}_V$.

Lemma 4.11. *Let (G, H, G') be a flight without any proper extensions, and suppose G is freely indecomposable, has no* QH *vertex groups, and that $\mathrm{JComp}(G) = \mathrm{JComp}(G')$. Let $\eta\colon G \to G'$ be the inclusion map. Then the following hold:*

- *For each vertex group W of the principle cyclic* JSJ *of H there are unique vertex groups V and $\eta_\#(V)$ of $\mathrm{JSJ}_H(G)$ and $\mathrm{JSJ}_H^*(G')$, respectively, such that $\nu(V) < W$, $\pi(W) < \eta_\#(V)$.*
- *W is obtained from V by adjoining roots to $\mathcal{E}_V$ and $\|\mathcal{E}_V\| \leq \|\mathcal{E}\| + 2\,\mathrm{b}_1(G)$.*
- *$\eta_\#(V)$ is obtained from $\pi(W)$ by adjoining roots to the images of $\mathcal{E}(V)$ (the edge groups incident to V)*
- *If π is injective on vertex groups then it is an isomorphism.*

The proof of Lemma 4.11 is contained in section 5, where graphs of spaces X_G, X_H, representing $\mathrm{JSJ}_H(G)$ and $\mathrm{JSJ}(H)$, respectively, and an immersion $X_G \to X_H$ representing $G \hookrightarrow H$, such that if the immersion is not one-to-one on edge spaces, then there must be a nontrivial extension, are constructed. The remainder of the lemma is largely formal, and relies on a simplification of the construction of strict homomorphisms from [Lou08b].

4.4. **Finishing the argument.** In this section we prove Theorem 1.4, postponing the proofs of lemmas used in the previous section until section 5. Let $(\mathcal{G}, \mathcal{H})$ be a staircase with complexity (b_0, d_0, e_0), such that no contraction has any proper extensions, and suppose that Theorem 1.4 holds for staircasess with complexity less than (b_0, d_0, e_0). By Theorem 3.6 there is some constant $B(b_0)$ such that $(\mathcal{G}, \mathcal{H})$ can be divided into $B(b_0)$ staircases of constant Scott complexity: (To maintain uniformity of the exposition, some sequences are allowed to be empty.)

$$(\mathcal{G}, \mathcal{H}) \mapsto \{(\mathcal{G}_i, \mathcal{H}_i)\}_{i=1,\dots,B(b_0,d_0)}$$
$$\mathcal{G}_i(1) = \mathcal{G}(j_i), \dots \quad \mathcal{H}_i(2) = \mathcal{H}(j_i + 1), \dots$$

Only the last of these can consist of freely indecomposable groups. Each staircase $(\mathcal{G}_i, \mathcal{H}_i)$, $i < B(b_0)$, by Theorem 4.8, has NInj bounded above by $b_0 \cdot \mathrm{NInj}(b_0 - 1, d_0, e_0)$, since there are at most b_0 freely indecomposable free factors.

Thus we may confine our analysis to freely indecomposable staircases. By Theorem 3.6, we may divide the staircase $(\mathcal{G}, \mathcal{H}, \mathcal{E})$ into boundedly many segments, the number depending only on the complexity of $\mathrm{b}_1(\mathcal{G})$, exhausting the tower, such that JComp is constant on each segment. By Lemma 4.3 we may assume that each segment is maximal with respect to $\leq_{\surd}$.

Like the case when each $\mathcal{G}(i)$ is freely decomposable, if $\mathcal{G}(i)$ has a QH vertex group, by Lemma 4.9 such staircases have bounded NInj.

The only possibility left is that the contractions of $(\mathcal{G}, \mathcal{H})$ are QH–free. Let I be the index set for $\mathcal{G}$, and color the triple $i < j < k$ red if $\mathrm{JSJ}^*_{\mathcal{H}(j)}(\mathcal{G}(j)) \cong \mathrm{JSJ}_{\mathcal{H}(k)}(\mathcal{G}(j))$, and blue otherwise. Then by Ramsey's theorem for hypergraphs, for all K there exists an L such that if $\|\mathcal{G}\| > L$ then there is a subset $I' \subset I$ of size at least K such that all triples whose elements are in I' have the same color.

Lemma 4.12. *There is an upper bound to the size of blue subsets which depends only on* $\mathrm{b}_1(\mathcal{G})$ *and* $\|\mathcal{E}\|$.

Proof. By Lemma 2.2, there are at most $2^{\|\mathcal{E}\|}$ equivalence classes of principle cyclic decomposition of G in which some element of $\mathcal{E}$ is elliptic. (There may be none.) Suppose $|I'| > 2^{\|\mathcal{E}\|}$, and consider the collection of principle cyclic decompositions $\left\{\mathrm{JSJ}_{\mathcal{H}(l)}(\mathcal{G}(i))\right\}$. Thus, for some $i < j < k$, $\mathrm{JSJ}_{\mathcal{H}(j)}(\mathcal{G}(i))$ and $\mathrm{JSJ}_{\mathcal{H}(k)}(\mathcal{G}(i))$ have the same elliptic subgroups. Then $\mathrm{JSJ}_{\mathcal{H}(k)}(\mathcal{G}(j)) \cong \mathrm{JSJ}^*_{\mathcal{H}(j)}(\mathcal{G}(j))$ since a JSJ decomposition is determined up to equivalence solely by its elliptic subgroups. $\square$

We are now on the home stretch. Suppose again that (b_0, d_0, e_0) is the lowest complexity for which Theorem 1.4 doesn't hold. By Lemma 4.12 and the prior discussion, there must be staircases $(\mathcal{G}, \mathcal{H}, \mathcal{E})$ of arbitrary NInj, which have complexity (b_0, d_0, e_0), are maximal, pure, and have no QH vertex groups.

Let $(\mathcal{G}, \mathcal{H})$ be such a staircase. Let $\mathcal{V}'_j(i)$ be the nonabelian vertex groups of $\mathcal{G}(i)$, indexed such that $\mathcal{V}'_j(i)$ maps to $\mathcal{V}'_j(k)$ for all $k > i$. Let $\mathcal{W}_j(i)$ be the corresponding rigid vertex group of $\mathcal{H}(i)$. By the second bullet of Lemma 4.11, $\mathcal{W}_j(i+1)$ is obtained from $\mathcal{V}'_j(i)$ by adjoining roots to $\mathcal{E}^{j,\prime}_i$, the set of elements of $\mathcal{E}_i$ which are conjugate into $\mathcal{V}'_j(i)$, along with the incident edge groups.

Let $\mathcal{V}_j(i) < \mathcal{V}'_j(i)$ be the image of $\mathcal{W}_j(i)$ in $\mathcal{G}(i)$. By the third bullet of Lemma 4.11, $\mathcal{V}'_j(i)$ is obtained from $\mathcal{V}_j(i)$ by adjoining roots to the images of the edge groups incident to $\mathcal{W}_j(i)$. Let

$$\mathcal{E}^j_i := \left\{E \cap \mathcal{V}_j(i) \mid E \in \mathcal{E}^{j,\prime}_i\right\} \cup \{E \cap \mathcal{V}_j(i) \mid E \in \mathcal{E}(\mathcal{V}_j(i))\}$$

The incident edge groups are cyclic, and we can build $\mathcal{W}_j(i)$ by simply adjoining roots to $\mathcal{E}^j_i$ in $\mathcal{V}_j(i)$. Then $\mathcal{E}^j_i$ is larger than $\mathcal{E}$ by at most the number of edge groups incident to $\mathcal{V}'_j(i)$, which is at most $2\,\mathrm{b}_1(\mathcal{G})$. That is,

$$\|\mathcal{E}^j_i\| \leq \|\mathcal{E}\| + 2\,\mathrm{b}_1(\mathcal{G}) \leq \|\mathcal{E}\| + 2\,\mathrm{b}_1(\mathcal{G})(\mathrm{depth}_{\mathrm{pc}}(\mathcal{H}) - \mathrm{depth}_{\mathrm{pc}}(\mathcal{W}_j))$$

Given a sufficiently long QH–free staircase $(\mathcal{G}, \mathcal{H}, \mathcal{E})$, we passed to a maximal extension (which we will also call $(\mathcal{G}, \mathcal{H}, \mathcal{E})$) of a substaircase of prescribed length,

such that the sequences of vertex groups $(\mathcal{V}_j, \mathcal{W}_j, \mathcal{E}^j)$ of the extension were cyclic staircases. The vertex groups of the extension are subgroups of the vertex groups of $\mathcal{H}$, hence the depth of $\mathcal{W}_j(i)$ is strictly less than the depth of $\mathcal{H}$. Moreover, the first betti number of $\mathcal{W}_j(i)$ is at most $\mathrm{b}_1(\mathcal{H})$ and by Lemma 4.11, $\mathrm{Comp}((\mathcal{G}, \mathcal{H}, \mathcal{E})) > \mathrm{Comp}((\mathcal{V}_j, \mathcal{W}_j, \mathcal{E}^j))$. There is an upper bound $B(b_0)$ to the number of vertex groups of the principle cyclic JSJ of a limit group with first betti number b_0. If $\|\mathcal{G}\| > B(b_0) \cdot \mathrm{NInj}(b_0, d_0 - 1, e_0 + 2b_0)$ there is some index l such that $\mathcal{H}(l) \twoheadrightarrow \mathcal{G}(l)$ is injective on all vertex groups. By the last bullet of Lemma 4.11, $\mathcal{H}(l) \twoheadrightarrow \mathcal{G}(l)$ is an isomorphism.

5. Hyperbolic to elliptic

5.1. **Graphs of spaces and immersions.** In this section we are given a fixed flight (G, G', H) of limit groups. By a *graph of spaces representing a principle cyclic decomposition* of a limit group G we mean a graph of spaces of the following form:

- For each rigid vertex group R a space X_R. Let $\mathcal{E}(R)$ be the edge groups incident to R, and for each $E \in \mathcal{E}(R)$ let $\sqrt{E}$ be the maximal cyclic subgroup of R containing the image of E. For each $E \in \mathcal{E}$ there is an embedded copy S_E of S^1 in X_R representing the conjugacy class of $\sqrt{E}$.
- For each edge E, a copy T_E of S^1, with basepoint b_E and an edge space $T_E \times \mathrm{I}$. On occasion we confuse T_E with $T_E \times \frac{1}{2}$, and sometimes refer to T_E as the edge space. The interval $b_E \times I$ is denoted t_E, and we choose an arbitrary orientation for t_E. The end of the edge space associated to E is attached via the covering map $T_E \looparrowright S_E$ representing $E \hookrightarrow \sqrt{E}$.
- For each abelian vertex group a torus T_A. If A is infinite cyclic then T_A has a basepoint b_A and the incident edge maps are simply covering maps which send b_E to b_A. These covering maps are isomorphisms unless the edge is adjacent to a QH vertex group, in which case they may be proper. For each edge space edge E adjacent to A, an edge space $T_E \times I$ and an embedded copy of T_E in T_A. This assumes edge groups not adjacent to QH vertex groups are primitive. Though there may be QH vertex groups, the cases which this definition is designed to handle do not, and we let this inconsistency slide. Unlike the rigid case, the embedded T_E need not be disjoint, though if they meet, they coincide. We require that any two embedded T_E, differ by an element of T_A, treated now as a group.
- For each QH vertex group Q a surface with boundary Σ_Q.
- If an edge group E is incident to a QH vertex group Q then T_E is identified with a boundary component of Σ_Q.
- The resulting graph of spaces has the fundamental group of G.

Let $\eta\colon G \hookrightarrow H$ be an inclusion of limit groups, and let Π_G and Π_H be principle cyclic decompositions of G and H, respectively, such that η maps vertex groups to vertex groups, edge groups to edge groups, and respects edge data, i.e., if $E \hookrightarrow V$, $\eta_\#(E) \hookrightarrow \eta_\#(V)$, then the obvious square commutes. If this is the case then η *respects* Π_G and Π_H. Let X_H be a graph of spaces representing Π_H. Then there

is a principle cyclic decomposition Π_G of G, a space X_G representing Π_G, and an *immersion* $\psi\colon X_G \to X_H$, inducing η, of the following form:

- For each abelian vertex group A there is a finite sheeted covering map $\psi|_{T_A}: T_A \looparrowright T_{\eta_\#(A)}$. The inclusions of incident edge spaces are respected by η:
$$\psi|_{\mathrm{Im}(T_E)} = (T_{\eta_\#(E)} \hookrightarrow T_{\eta_\#(A)}) \circ \psi|_{T_E}$$
- For each E there is a finite sheeted product-respecting covering map $T_E \times \mathrm{I} \looparrowright T_{\eta_\#(E)} \times \mathrm{I}$ which maps t_E to $t_{\eta_\#(E)}$. If E is adjacent to a QH vertex group then the degree of the covering map is one.
- For each R there is a map $X_R \to X_{\eta_\#(R)}$ such that for each edge group E incident to R the following diagram commutes:
$$\begin{tikzcd} T_E \times \{0\} \arrow[r] \arrow[d] & X_R \arrow[d] \\ T_{\eta_\#(E)} \times \{0\} \arrow[r] & X_{\eta_\#(R)} \end{tikzcd}$$
  Likewise for $\times \{1\}$.
- For each Σ_Q a homeomorphism $\Sigma_Q \to \Sigma_{\eta_\#(Q)}$. The maps $X_R \to X_{\eta_\#(R)}$ (similarly for T_A's) respect attaching maps of boundary components of surfaces.
- If E_1 and E_2 are incident to A and T_{E_1} and T_{E_2} have the same image in T_A, then $\eta_\#(E_1) \neq \eta_\#(E_2)$.

The existence of immersions as above is an easy variation on Stallings's folding. One way to construct immersions of graphs representing subgroups is to pass to the cover of a graph representing a subgroup and trimming trees. There is an analogous construction in this context.

5.2. **Roots, immersions, and resolving.** We need to be able to represent conjugacy classes of elements of limit groups as nice paths in graphs of spaces.

Definition 5.1 (Edge path)**.** Let X_G be a graph of spaces representing a principle cyclic decomposition of G. The *zero skeleton* of X_G, denoted X_G^0, is the union of vertex spaces.

An *edge path* in a graph of spaces X_G is a map $p\colon [0,1] \to X_G$ such that $p^{-1}(X_G^0)$ contains $\{0,1\}$ and is a disjoint collection of closed subintervals. Let $[a,b]$ be the closure of a complementary component of $p^{-1}(X_G^0)$. Then p maps $[a,b]$ homeomorphically to some t_E.

Let X_R be a vertex space. Set ∂X_R be the union of copies of edge spaces contained in X_R. An edge path p is reduced if every restriction $p|_{[a,b]}([a,b]\,;\{a,b\}) \to (X_R, \partial X_R)$ does not represent the relative homotopy group $\pi_1(X_R, \partial X_R)$

A continuous map $\gamma\colon \mathrm{S}^1 \to X_G$ is *cyclically reduced* if all edge-path restrictions of γ to subintervals $\mathrm{I} \subset \mathrm{S}^1$ are reduced edge paths.

The following lemma is standard and follows easily from Stallings folding [Sta65, Sta83] and the definitions.

Lemma 5.2. *If $g \in G$ then there is a cyclically reduced edge path $\gamma\colon \mathrm{S}^1 \to X_G$ representing the conjugacy class $[g]$.*

Let $\psi\colon X_G \looparrowright X_H$ be an immersion representing $G \hookrightarrow H$. If $\gamma\colon \mathrm{S}^1 \to X_G$ is a reduced edge path then $\psi \circ \gamma$ is a reduced edge path in X_H.

For each edge E of X_G, we introduced a subset t_E of the edge space $T_E \times I$. We think of t_E as a formal element representing the path $I \to b_E \times \mathrm{I}$ with a fixed but arbitrary orientation. Let $\tau(t_E)$ be the image of the basepoint of T_E in the vertex space of X_G at the terminal end of $T_E \times \mathrm{I}$, and let $\iota(t_E)$ be the image of the basepoint in the vertex space at the initial end of T_E. Then every nonelliptic element represented by a cyclically reduced path can be thought of as a composition t_E's, their inverses, and elements of relative homotopy groups of vertex spaces. Moreover, if the subword $t_E g t_E^{-1}$ appears then g is not contained in the image of E.

Let $\gamma \in G$ be represented by a cyclically reduced edge path γ; $\psi \circ \gamma$ is an edge path in X_H, and if it is not cyclically reduced, then for some sub-path $t_E h t_E^{-1}$ of γ (we may need to reverse the orientation of t_E), the image of this subpath is homotopic into $\eta_\#(T_E)$, which means that $[h] \in \eta_\#(E)$. Since γ is reduced, $[h] \notin E$, and since the image of E in $\eta_\#(E)$ is finite index, for some $l > 0$, $[h]^l \in E$. Since edge groups are primitive unless adjacent to QH vertex groups, E must be attached to a boundary component of a QH vertex. This implies that $\eta_\#(E)$ is also attached to a boundary component of a QH vertex group, but this means $E \to \eta_\#(E)$ is an isomorphism, contradicting the fact that $[h] \notin E$.

Let G and H be freely indecomposable limit groups, H obtained from G by adjoining roots to $\mathcal{E}$, $\eta\colon G \hookrightarrow H$. Let Π_G and Π_H be principle cyclic decompositions and suppose that if K is elliptic in Π_G if and only if $\eta(K)$ is elliptic in Π_H. Let $\psi\colon X_G \to X_H$ be an immersion representing the inclusion.

Without loss of generality, suppose that all elements of $\mathcal{E}$ are self-centralized and nonconjugate. Let $\mathcal{E}_e$ be the elements of $\mathcal{E}$ which are elliptic in Π_G and let $\mathcal{E}_h$ be the elements of $\mathcal{E}$ which are hyperbolic in Π_G.

For each $E \in \mathcal{E}$ let T_E be a torus representing E, $T_{F(E)}$ a torus representing $F(E)$, and let $T_E \to T_{F(E)}$ be the covering map corresponding to the inclusion $E \hookrightarrow F(E)$. Let M_E be the mapping cylinder of the covering map. If $\langle\gamma\rangle \in \mathcal{E}$ we abuse notation and refer to $M_{\langle\gamma\rangle}$ as M_γ. The copy of $T_{F(E)}$ in M_E is the *core* of M_E, and if E is infinite cyclic, it is the *core curve*. The copy of T_E in M_E is the *boundary*, and is denoted ∂M_E.

For each element $E \in \mathcal{E}_e$, let $f_E\colon T_E \to X_G$ be a map representing the inclusion $E \hookrightarrow G$ which has image in a vertex space of X_G. If E is an abelian vertex group of Π_G then we identify T_E with the torus $T_A \subset X_G$. For each $\langle\gamma\rangle \in \mathcal{E}_h$,[3] represent γ by a reduced edge path, abusing notation, $\gamma\colon \partial M_\gamma \to X_G$.

Build a space X'_G by attaching the M_E and M_γ to X_G along T_E and $\mathrm{Im}(\gamma)$ by the maps f_E and γ, respectively.

By hypothesis there is a π_1–surjective map $\psi'\colon X'_G \to X_H$. We choose this map carefully: For $E \in \mathcal{E}_e$, $F(E)$ has elliptic image in Π_H. Choose a map $T_{F(E)} \to$

[3]All elements of $\mathcal{E}_h$ are infinite cyclic.

X_H with image contained in the appropriate vertex space of X_H, and extend the map across M_E so that M_E also has image contained in the vertex space of X_H. For $\langle\gamma\rangle \in \mathcal{E}_h$, the core curve of M_γ is a k_γ–th root of γ. Choose a cyclically reduced representative of $\sqrt[k]{\gamma}\colon S^1 \to X_H$ and let the map on the core curve agree with this representative.

The restriction of ψ', defined thus far, to the disjoint union of X_G and the core curves of the M_γ, is transverse to the subsets $T_{\eta_\#(E)} \times \left\{\frac{1}{2}\right\}$. Extend ψ' to X_G so the composition $M_\gamma \hookrightarrow X'_G \xrightarrow{\psi'} X_H$ is transverse to all $T_{\eta_\#(E)} \times \frac{1}{2}$. Let Λ be the preimage

$$\psi'^{-1}\left(\sqcup_{E\in\mathcal{E}(G)}\left(T_{\eta_\#(E)} \times \left\{\frac{1}{2}\right\}\right)\right)$$

Suppose some component of Λ is a circle which misses the boundary and core of some M_γ. By transversality this component of Λ is a one manifold without boundary, and is therefore a circle. If this circle bounds a disk then there is a map homotopic ψ', which agrees with ψ' on the core curves and X_G such that the number of connected components of the preimage is strictly lower. If the circle doesn't bound a disk in M_γ then it is boundary parallel. If this is the case then γ is elliptic and we have a contradiction.

Fix a mapping cylinder M_γ and consider the preimage of Λ under the map $M_\gamma \to X'_G$. The preimage is a graph all of whose vertices are contained in the core curve of M_γ or in the boundary of M_γ. If any component of the preimage of Λ doesn't connect the boundary of M_γ to the core curve, then it is an arc and there is an innermost such arc which can be used to show that one of either γ or γ' is not reduced. Thus the preimages of arcs connect the core curve to the boundary.

Let b be a point of intersection of Λ with the core curve of M_γ. There are k_γ arcs, where k_γ is the degree of the root added to γ, $s_1, \ldots, s_{k_\gamma}$ (cyclically ordered by traversing ∂M_γ) in Λ connecting b to ∂M_γ. Now consider the arcs as paths $s_j\colon [0,1] \to M_\gamma$. The composition $p_\gamma := s_2^{-1}s_1$ is a path in M_γ from ∂M_γ to ∂M_γ. Let D_γ be the sub-path of γ obtained by traversing ∂M_γ from $* := s_1 \cap \partial M_\gamma$ to $*_2 := s_2 \cap \partial M_\gamma$. The path $D_\gamma p_\gamma$ is homotopic, relative to $*$, to $s_1^{-1}\sqrt[k]{\gamma}s_1$. In particular,

$$(D_\gamma p_\gamma)^{k_\gamma} \simeq \gamma$$

A possible neighborhood of a component of Λ is illustrated in Figure 3.

Three interrelated lemmas.

Lemma 5.3. *Suppose $\eta\colon G \hookrightarrow H$, H obtained from G by adjoining roots to $\mathcal{E}$, G freely indecomposable. Let Π_H be a one-edged splitting of G over a cyclic edge group E_H. Let Π_G be the splitting G inherits from its action via η on the Bass-Serre tree for Π_H. Represent Π_H by a graph of spaces X_H, and choose a graph of spaces X_G and an immersion $\psi\colon X_G \looparrowright X_H$ representing η. Suppose that Π_G is one-edged, and that the edge group is E. If $\mathcal{E}_h$ is nonempty then $E \hookrightarrow \eta_\#(E)$ is a* proper *finite index inclusion.*

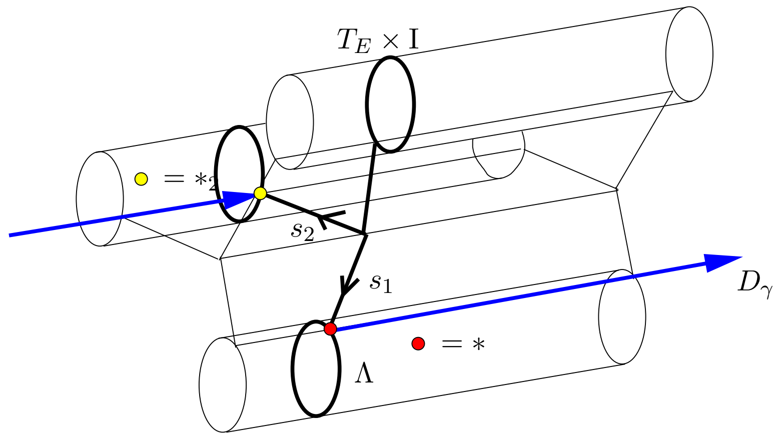

FIGURE 3. A neighborhood of a component of Λ in X'_G

Proof. Let $\langle\gamma\rangle \in \mathcal{E}_h$, and represent γ by a reduced edge path crossing t_E. Since ψ is one-to-one on edge spaces, p_γ is a closed path. As such, it represents an element of the fundamental group of X'_G. Then $[\psi \circ p_\gamma] \in \eta_\#(E)$. If $[\psi \circ p_\gamma] \in \mathrm{Im}(E)$ then there is a path p'_γ in T_E which is homotopic in $T_{\eta_\#(E)}$, relative to the image of $*$, to $\psi \circ p_\gamma$. Let $\alpha = D_\gamma p'_\gamma$. Then $\psi \circ \alpha$ is homotopic rel the image of $*$ to $\psi \circ D_\gamma p_\gamma$. But then $[\alpha]^{k_\gamma} = \gamma$ contradicting indivisibility of γ. □

Lemma 5.4. *Let $G \hookrightarrow G'$ be an adjunction of roots. Let $\Pi_{G'}$ be a principle cyclic splitting of G' with one abelian vertex group A, let Π_G be the associated splitting of G, and represent $G \hookrightarrow G'$ by an immersion $\eta\colon X_G \looparrowright X_{G'}$ reflecting Π_G and $\Pi_{G'}$. Suppose there is a unique vertex group A' of Π_G mapping to A, and that there is at most one element of $\mathcal{E}$ conjugate into A'. If η is one-to-one on edges adjacent to A' then the induced map $F(A') \to A/P(A)$ is onto.*

Proof. Let $E_1, \ldots, E_n$ be the edges adjacent to A', and set $F(E_i) = \eta_\#(E_i) = P(A)$. Let H be the limit group defined as follows: Let $\Delta = \Delta(R_j, E_i, A')$ be a graphs of groups representation of Π_G. Let $\mathcal{E}_{R_j}$ be the subcollection of $\mathcal{E}$ consisting of elements conjugate into R_j. Let

$$S_j := \mathrm{Im}_{G'}\left(\langle R_j\left[\sqrt{\mathcal{E}_{R_j}}\right], gF(E_i)g^{-1}\rangle_{gE_ig^{-1}<R_j}\right)$$

and

$$A'' := \mathrm{Im}_{G'}(F(A'), P(A))$$

Let $H := \Delta(S_j, F(E_i), A'')$ There are maps $G \hookrightarrow H \hookrightarrow G'$. We now show that $H \hookrightarrow G'$ is actually surjective. To do this we need to show that every element $\langle\gamma\rangle$ of $\mathcal{E}_h$ has a k_γ–th root in H. This is precisely the argument given at the end of Lemma 5.3. Let $G' \twoheadrightarrow A/P(A)$ be the map which kills all vertex, edge groups, and stable letters, other than A. The quotient map clearly kills everything except A and $F(A')$, giving the desired surjection. □

Lemma 5.5. *Let (G, H, G') be a flight without any proper extensions. Suppose G is freely indecomposable, has no* QH *vertex groups, and* $\mathrm{JComp}(G) = \mathrm{JComp}(G')$. *Represent the $G \hookrightarrow H$ by an immersion $X_G \looparrowright X_H$, representing* $\mathrm{JSJ}_H(G)$, *and* $\mathrm{RJSJ}(H)$, *respectively. Then ν is one-to-one on edge spaces.*

Every vertex group W of H is obtained from a vertex group V of $\mathrm{JSJ}_H(G)$ *by adjoining roots to the elements of $\mathcal{E}$ which are conjugate into V, along with edge groups incident to V.*

Proof of Lemma 5.5. Represent $G \hookrightarrow H$ by an immersion $X_G \to X_H$, where X_G represents $\mathrm{JSJ}_H(G)$ and X_H represents the principle cyclic JSJ of H. For each edge E_i of X_G let e_i be a generator, let k_i be the largest degree of a root of e_i in H, let $F(E) = \langle f_i \rangle$, and let $E \hookrightarrow F(E)$ be the map which sends e_i to $f_i^{k_i}$. Let $\mathcal{E}'$ be the collection of elements of $\mathcal{E}$ which are elliptic in H along with all edge groups of $\mathrm{JSJ}_H(G)$.

Consider the group $G\left[\sqrt{\mathcal{E}'}\right]$. Let $\Delta = \Delta(R_i, A_j, E_k)$ be a graph of groups representation of $\mathrm{JSJ}_H(G)$. Let $\mathcal{E}_{R_i}$ be the set of elements of $\mathcal{E}'$ which are conjugate into R_i. Likewise, let $\mathcal{E}_{A_j}$ be the set of elements of $\mathcal{E}'$ which are conjugate into A_j. Let

$$S_l = \mathrm{Im}_H(\langle S_l, gBg^{-1}\rangle_{gBg^{-1}<S_l, B\in\mathcal{E}_{S_l}})$$

where $gBg^{-1} < S_l$, and where S_l is either some rigid vertex group R_i or abelian vertex group A_j. Let

$$H' = \Delta(R'_l, A'_j, F(E_k))$$

and choose a graph of spaces $X_{H'}$ representing this decomposition of H'. There is a pair of maps of graphs of spaces $X_G \to X_{H'}$, $X_{H'} \to X_H$, and there is an epimorphism $G\left[\sqrt{\mathcal{E}'}\right] \twoheadrightarrow H'$. The map $\psi' \colon X_G \to X_{H'}$ is one-to-one on edge spaces. Moreover, H' is a limit group since the map $H' \to H$ is clearly strict.

The proof of the lemma will be complete if we can show that ψ' extends to X'_G, that is, if H' contains all roots of elements adjoined to $\mathcal{E}$. Then the image of $G\left[\sqrt{\mathcal{E}}\right]$ (with the induced graph of groups decomposition) in H' is a nontrivial extension of H.

Consider the paths D_γ and p_γ defined previously through resolving. We defined $p_\gamma := s_2^{-1}s_1$ and set $* = s_1 \cap \partial M_\gamma$. Let $*_2 := s_2 \cap \partial M_\gamma$. To show that H' has a k_γ–th root of γ we need to show that $X_{H'}$ has a path p'_γ from the image of $*_2$ to the image of $*$ whose image under $X_{H'} \to X_{G'}$ is homotopic rel endpoints to the image of p_γ.

Suppose that $T_{E_1} \times \frac{1}{2}$ and $T_{E_2} \times \frac{1}{2}$ are the midpoints of edge spaces containing $*$ and $*_2$, respectively, and suppose, without loss of generality, that D_γ starts and ends by traversing the second and first halves of T_{E_1} and T_{E_2}, respectively, in the positive direction. The first key observation to make is that we can choose the orientations of t_{E_i} so that the terminal endpoints of t_{E_1} and t_{E_2} are both contained in some T_A: E_1 and E_2 are conjugate in H, must therefore be conjugate in G since $\mathrm{JComp}(G') = \mathrm{JComp}(G)$, and cannot both be adjacent to a rigid vertex group of G, otherwise there is a rigid vertex group R of G such that $v(R) > v(\eta_\#(R))$.

The only other possibility is that they are both adjacent to an abelian vertex group A, as claimed.

Let $t^+_{\varphi_\#(E_i)}$ be the half of $t_{\varphi_\#(E_i)}$ obtained by traversing $t_{\varphi_\#(E_i)}$ from the midpoint to the terminal endpoint. By Lemma 5.4, $H' \to H$ is surjective on abelian vertex groups, and by construction, the terminal endpoints of $t^+_{\varphi_\#(E_i)}$ agree. Let $p''_\gamma := t^+_{\varphi_\#(E_2)}(t^+_{\varphi_\#(E_1)})^{-1}$. Then p''_γ is a path from $\varphi(*_2)$ to $\varphi(*)$ whose image in X_H is homotopic rel endpoints into $\psi_\#(E_1)(=\psi_\#(E_2))$. Since $H' \twoheadrightarrow H$ is surjective on edge groups, there is a closed path h_γ in $(T_{\varphi_\#(E_1)}, \varphi(*))$ which maps to the image of p_γ. Set $p'_\gamma := h_\gamma p''_\gamma$. The image of p'_γ is homotopic rel endpoints to the image of p_γ in X_H. Arguing as in Lemma 5.3, $(\varphi \circ D_\gamma)p'_\gamma$ is a k_γ–th root of $\varphi \circ \gamma$ and the map $X'_G \to X_H$ factors through $X_{H'}$.

Thus there is a map $X'_G \to X_{H'}$. Since H has no proper extensions, $\mathrm{Im}_{H'}\left(G\left[\sqrt{\mathcal{E}}\right]\right) \to H$ is an isomorphism. In particular, $X_G \to X_H$ is one-to-one on edges and the situation above never occurs.

Consider the construction of H'. Now that we know that $H' \cong H$, Δ *must* be the principle cyclic JSJ of H. If there is a principle cyclic splitting of H not visible in Δ then it must be a cyclic splitting inherited from the relative (to incident edge groups) principle cyclic JSJ decomposition of some vertex group of Δ. On the other hand, all vertex groups of Δ must be elliptic in the principle cyclic JSJ of H since they are obtained by adjoining roots to subgroups of G which are guaranteed to be elliptic in the principle cyclic JSJ of H. $\square$

This nearly completes the proof of Lemma 4.11. We need to prove that the vertex groups of $\mathrm{JSJ}^*_H(G')$ are obtained from the images of the vertex groups of $\mathrm{JSJ}(H)$ by adjoining roots, and that π is injective if its restrictions to vertex groups are injective.

Let $\Delta = \Delta(R_i, A_j, E_k)$ be a graph of groups decomposition representing the principle cyclic JSJ of H. We know that all vertex and edge groups of Δ map to vertex and edge groups of $\mathrm{JSJ}^*_H(G')$. Let $\Phi_s(\pi)\colon \Phi_s(H) \twoheadrightarrow G'$ be the strict homomorphism constructed in [Lou08b, § 5], and also in the appendix of this paper, and let $\Phi_s(\Delta)$ be the principle cyclic decomposition of $\Phi_s(H)$ in which all images of vertex groups of Δ are elliptic. Clearly $\Phi_s(\pi)$ maps elliptic subgroups of $\Phi_s(\Delta)$ to elliptic subgroups of $\mathrm{JSJ}^*_H(G')$. Moreover, if A is a noncyclic abelian vertex group of H, then by construction, $\Phi_s(\pi)$ maps $A/P(A)$ onto $\pi_\#(A)/P(\pi_\#(A))$. Thus all modular automorphisms of $\Phi_s(H)$ supported on abelian vertex groups of Δ push forward to modular automorphisms of G'. Another consequence of the hypothesis $\mathrm{JComp}(G) = \mathrm{JComp}(G')$ is that $\Phi_s(H) \to G'$ is one to one on the set of edge groups adjacent to every vertex group, hence every Dehn twist of $\Phi_s(H)$ pushes forward to a Dehn twist of G'. A strict map which allows all modular automorphisms to push forward is an isomorphism, therefore $\Phi_s(\pi)$ is an isomorphism.

The third bullet follows immediately from the construction of Φ_s.

APPENDIX A. CONSTRUCTING STRICT HOMOMORPHISMS

We give here a description of the process of constructing strict homomorphisms of limit groups. Let G be a group with a one-edged splitting Δ with nonabelian vertex groups of the form $G \cong R *_{\langle e \rangle} S$, and suppose there is a map $\varphi \colon G \to L$, L a limit group, which embeds R and S. Then R and S are limit groups. Suppose further that φ embeds $R *_{\langle e \rangle} Z_S(\langle e \rangle)$ and $Z_R(\langle e \rangle) *_{\langle e \rangle} S$. Then φ is *strict*, and G is a limit group. There is a process, whose output is a limit group $\Phi_s(G)$, which takes the data (G, Δ, L, φ) and produces a triple $G \to \Phi_S(G) \to L$, such that the composition is φ, $\Phi_S(G)$ splits over the centralizer of $\langle e \rangle$, and $\Phi_S(G) \to L$ is strict.

The process is one of pulling centralizers and passing to images of vertex groups in a systematic way. The reader should compare this to the more general construction detailed in [Lou08b], and a formally identical version in the proof of [BF03, Lemma 7.9]. Let $G = G_0$. Define for

- odd i: $G_i = R_{i-1} *_{Z_{R_{i-1}}(\langle e \rangle)} S_i$, where
$$S_i := \mathrm{Im}_L(Z_{R_{i-1}}(\langle e \rangle) *_{Z_{S_{i-1}}(\langle e \rangle)} S_{i-1})$$
- even i: $G_i = R_i *_{Z_{S_{i-1}}(\langle e \rangle)} S_{i-1}$, where
$$R_i := \mathrm{Im}_L(R_{i-1} *_{Z_{S_{i-1}}(\langle e \rangle)} Z_{S_i}(\langle e \rangle))$$

We claim that this process terminates in finite time. The sequence of quotients $G_0 \twoheadrightarrow G_1 \twoheadrightarrow \dots$ embeds edge groups at every step. Since abelian subgroups of limit groups are finitely generated and free, and since finitely generated free abelian groups satisfy the ascending chain condition the assertion holds. The direct limit G_∞ is called $\Phi_s(G)$.

This discussion is relevant to the proof of Lemma 5.5, but we must vary the construction a little. Let $\bar{H}$ be the quotient of H obtained by passing to the images in G' of vertex groups of the (restricted) principle cyclic JSJ of H, with the induced graph of groups decomposition $\Delta(\bar{R}_i, A_j, E_k)$. The *core* of $\bar{H}$, $\mathrm{Core}(\bar{H})$ is the group obtained by replacing each abelian vertex group A by its peripheral subgroup. Consider the situation in Lemma 5.5. There is a homomorphism $\mathrm{Core}(\bar{H}) \to G'$, and each group is equipped with a principle cyclic decomposition $\Delta_{\mathrm{Core}(\bar{H})}$ and $\Delta_{G'}$, respectively. Moreover, the nonabelian vertex groups of $\Delta_{\mathrm{Core}(\bar{H})}$ map to nonabelian vertex groups of G', and the edge groups of $\mathrm{Core}(\bar{H})$ map to edge groups of $\Delta_{G'}$. The centralizers of edges incident to nonabelian vertex groups of G' are infinite cyclic, and this implies that in the process of pulling centralizers in $\mathrm{Core}(\bar{H})_i$, the pulled group is always infinite cyclic. Each vertex group of $\mathrm{Core}(\bar{H})_i$ has elliptic image in G', and since G' is principle, centralizers are cyclic in the relevant vertex groups of G'. Iteratively adjoining roots to an infinite cyclic subgroup and passing to quotients multiple times can be accomplished in one step, thus the vertex groups of $\mathrm{Core}(\bar{H})_\infty$ are obtained from the vertex groups of $\mathrm{Core}(\bar{H})$ by adjoining roots to incident edge groups. There are surjective maps $H \twoheadrightarrow \Phi_s(H) := \mathrm{Core}(\bar{H})_\infty *_{Z(P(A_j))} (Z(P(A_j)) \oplus A/P(A_j)) \twoheadrightarrow L$.

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